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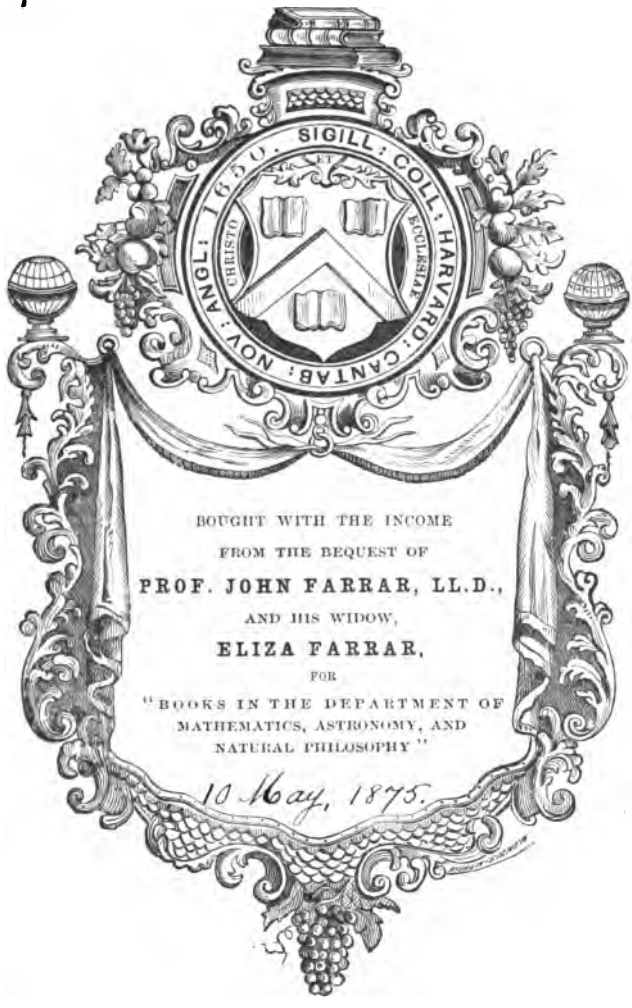
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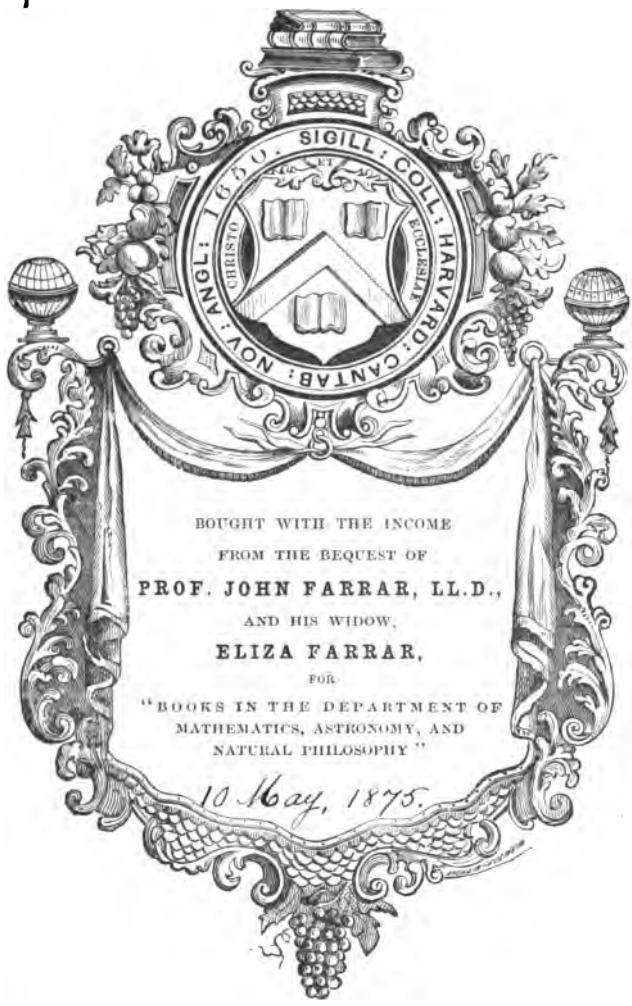


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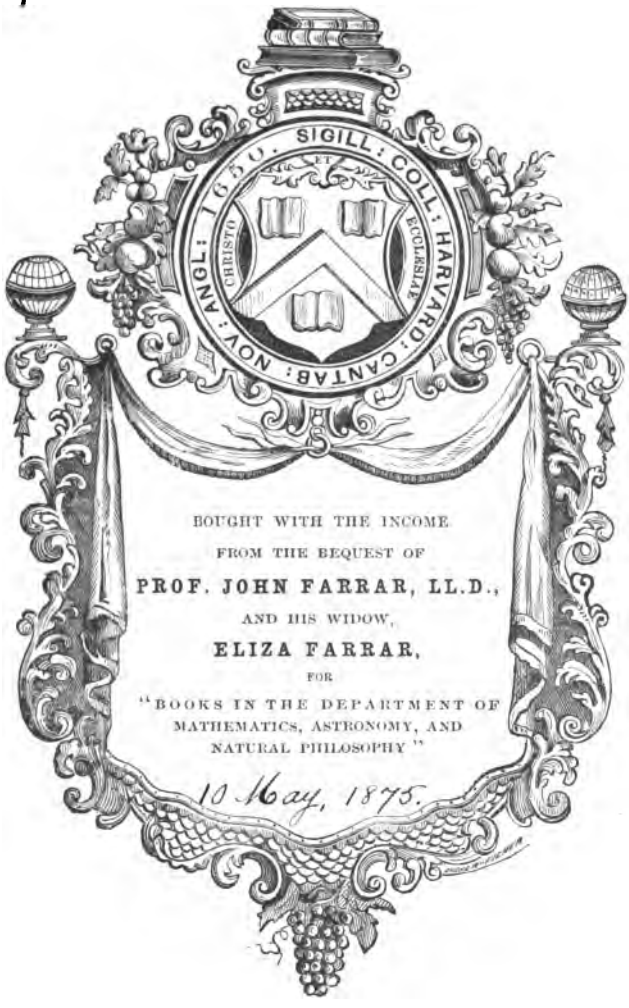
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CHAPTER I.

1. MECHANICS is the science which treats of the effects produced by force acting on material bodies. When the different forces acting on any body counteract each other so that the body remains at rest, the forces are said to be in equilibrium. The consideration of the conditions to which the forces must be subject that this may be the case, is the object of the science of Statics.

When these conditions are not fulfilled by the forces, the body will not remain at rest. The investigation of the laws which regulate the motion which takes place, and of the nature of the motion itself, will form the subject of the present treatise.

Before we proceed, however, there are certain ideas of which it is necessary to form an accurate conception; and, in order to be definite, we will at present confine our attention to bodies so small that they may without sensible error be treated as geometrical points, but we must at the same time consider them to be possessed of all the properties of matter. This is what must be understood when particles, or material points, or molecules are spoken of.

2. It would be useless to attempt to define space and time. No explanation could in any way render the ideas clearer. The measures of them, on the contrary, require the greatest degree of attention. In Dynamics we are only concerned with linear space, or length, or distance, and this is always to be understood when the term *space* is used. Now every concrete magnitude must be measured by some definite magnitude of the same kind; thus, space must be measured by space, time by time.

With respect to space nothing can be easier. We can fix on some definite length, and can take a rule or a piece of string of that length, and apply it over and over again to the space which we wish to measure to find how often it is contained in it; and we consider that we know the magnitude of that space when we know the number of times that our measure is contained in it. Thus, if it be the distance between two points A and B that we wish to determine, we consider that we know it when we have found that our measure will go in it a times, a being whole or fractional; and we say that

$$AB = a \text{ times the measure,}$$

or more commonly, suppressing the measure, we say that $AB = a$. In this case our measure is called the "unit of length," and the space AB is said to be numerically represented by a. It is quite obvious that the greater or less our measure or unit is chosen, the less or greater will be the numerical representation of any given space AB. The choice of this unit is perfectly arbitrary; but we must always bear in mind that it is a space.

3. We cannot apply the same method of proceeding to "time:" for we are not able to take an interval of time and apply it to other intervals to determine how often it is contained in them. An instant of time, corresponding to a point in space, can only be marked by the occurrence of some event. Now we can easily conceive a number of events happening one after the other in such a manner that the interval between any two shall be of the same duration as the interval between any other two. It is possible and easy to conceive, *a priori*, this equality of intervals, but it is impossible, *a priori*, to fix on any series of events which we can be sure will so occur at equal intervals. We shall see, when we have advanced somewhat further, that such a series of events can be procured, but till then we must rest satisfied with being able to conceive such a series. The possibility or impossibility of such a series existing will in no wise affect the accuracy of the notion of equal times. If we take one of these intervals as our measure, we shall have any interval which lasts during n of those intervals expressed by n times the interval, where n may be whole or fractional. And after this interval is agreed on, we consider the length of any other interval determined when n is known, and we express it simply by n: in other words, we take the agreed-on interval as our unit. The duration of this "unit of time" is perfectly arbitrary, and as it is greater or less, n the numerical representation of any proposed interval will be less or greater.

4. Having now explained the method of measuring time and space, and of representing them numerically, we come to the motion of a material point. The simplest kind of motion that we can conceive is what is called "uniform rectilinear motion." When a point moves so that at every instant during the whole duration of the motion it is found in the same straight line, the motion is said to be rectilinear. If, moreover, it moves so as always to pass over equal spaces in equal intervals of time, at what part soever of the motion those intervals be taken, or however long or short their duration, then the motion is said to be uniform. Hence if it pass over a space s in one interval, it would pass over a space $2s$ in two such intervals, and a space ns in n such intervals.

5. Like space and time "velocity" is a term which cannot be defined. We may say it is the "rate" of motion or the "pace" or "speed," but we are only giving other words of the same meaning, not explaining the idea. We can however, as in those cases, explain accurately how it is measured.

Suppose two points to be moving uniformly as described above, and suppose that the space which one describes in any interval of time is equal to the space which the other describes in the same interval, then the velocities of these two points are the same.

Now before we can express velocity numerically, it is necessary to know what is meant when one velocity is said to be twice another. We saw in the case of space, that if we took any given length, and at the end of that took another length equal to the former, the whole length so taken was twice the original length.

Similarly, if we supposed an interval of any definite duration, and then at the end of it another interval of the same duration, the whole interval would be twice one of the first intervals.

We cannot however adopt the same method with velocity, we cannot form any distinct conception of what is meant by one velocity being the sum of two other velocities without bringing to our aid ideas which tacitly assume the point in question. The common notion, however, is very simple and sufficiently exact.

Generally one thing is said to move twice as fast as another, or to have twice the velocity of another when it passes over twice the space in the same time. This is the foundation of the method of representing velocity numerically. And this definition of double velocity, or of "twice as fast," is at once applicable to any multiple of the velocity. Thus, if a particle moving with a given velocity passes over in a given time a space s (that is, a space whose nume-

rical representation is s), a particle moving with twice that velocity would pass over in the same interval of time a space $2s$; and a particle moving with n times the given velocity would in the same interval pass over a space ns. If the first-mentioned velocity be that chosen as a unit, the latter velocity would be represented numerically by n.

The definition just given, of one velocity being n times another, is also expressed by saying that the velocity varies as the space described in a given interval of time.

Now suppose a point moving with a velocity represented as above explained by v to pass over the space s in a given interval of time t, and suppose another point to move so as to pass over an equal space s in an n^{th} part of that interval of time, then this latter point would pass over a space ns in this interval of time t, and therefore its velocity would be represented by nv.

This is expressed by saying that when two points describe equal spaces their velocities are inversely as the times of describing them.

If a point describes uniformly a known space in a known time, the velocity of that point is known as much as we can conceive a velocity to be. Suppose then that the velocity by which we measure other velocities, or which we take as our unit of velocity, is that with which a point would move through a space s_1 in a time t_1. And suppose that a body moving with v' times this velocity would describe a space s in the same time t_1, then we have seen that

$$s = v' . s_1.$$

And suppose that a body moving with v'' times this last velocity would move over the space s in time t, then we have seen that

$$t = \frac{t_1}{v''}.$$

Also this last velocity is $v'v''$ times the velocity which we have taken as our measure. Let $v'v'' = v$, then the last velocity is v times our measure, and with it our point moves over the space s in the time t. And referring to our equations we see that

$$v = v'v'' = \frac{s}{s_1} . \frac{t_1}{t} = \frac{s}{t} . \frac{t_1}{s_1}.$$

If we choose as our measure such a velocity that the ratio $\frac{t_1}{s_1}$ is unity, then we have $v = \frac{s}{t}$.

This will be the case if we have $t_1 = 1$ and $s_1 = 1$, which is the same thing as taking for our measure the velocity with which a body would uniformly describe the unit of space in the unit of time.

This for the future we shall always do. In that case the equation $v=\frac{s}{t}$, which we shall consider as our fundamental equation, expresses that the numerical representation of the velocity of a point is the fraction whose numerator and denominator are respectively the numerical representations of the space described and of the time in which it is described.

6. We may observe that the particular length which we fix on as our unit or measure of space, and the particular interval whose duration we fix on as the unit or measure of time, are both perfectly arbitrary and perfectly independent, but that when we have fixed on them, we shall have fixed on the particular velocity which we take as our unit of velocity. In different investigations different units of time and space are chosen, but it is obviously essential that throughout the same investigation the same units should be adhered to.

7. We have hitherto considered the case of a material point moving uniformly, and have seen how its velocity is measured and numerically represented, and have obtained our fundamental equation $v=\frac{s}{t}$, which gives the same numerical value to v however large or small t be taken. We will now consider the case of a body whose velocity is changing from one instant to another.

The first question that arises is, "What do we mean when we speak of the velocity of the point at any instant?" A little consideration however will shew that our idea of velocity is independent of the supposition of its being uniform. We can suppose a body moving uniformly to be moving just as fast as another body, whose motion is not uniform, was at some particular instant: in other words, that at some particular instant their velocities were equal.

If then at any instant we suppose the motion of a body to become uniform, the velocity at that instant remaining unaltered, the numerical representation of this uniform velocity will be the numerical representation of the variable velocity at that instant.

Suppose now for distinctness that the velocity of a body at a certain instant is increasing, and that its numerical representation estimated as we have explained is v; and suppose that after an interval δt has elapsed the velocity is $v+\delta v$, and that the space which has been described in that time is δs.

(The expressions δt, δv, δs must be understood as single symbols expressing no more than we have hitherto done by t, v, and s, that is, they represent numbers, and are used as single symbols or single letters.)

Also let δs_1 be the space that would have been described in the time δt if the body had moved uniformly during that time with the velocity v, and δs_2 the space that would have been described in the same time δt had the body moved uniformly with the velocity $v + \delta v$. Then, from what has been shewn before, we have

$$v = \frac{\delta s_1}{\delta t}, \text{ and } v + \delta v = \frac{\delta s_2}{\delta t}.$$

Now the space δs which is actually described will be greater than the space δs_1 and less than the space δs_2. Therefore we shall have

$$\text{the fraction } \frac{\delta s}{\delta t} \text{ greater than } \frac{\delta s_1}{\delta t},$$

$$\text{and less than } \frac{\delta s_2}{\delta t},$$

that is, we shall have the fraction

$$\frac{\delta s}{\delta t} \text{ greater than } v \text{ and less than } v + \delta v,$$

and this is true however small δt be taken; but by diminishing δt we may make δv less than any assignable quantity, and δs will also be diminished at the same time; if, therefore, in accordance with the notation of the differential calculus, we represent by $\frac{ds}{dt}$ the limit to which the fraction $\frac{\delta s}{\delta t}$ approaches when δt is diminished indefinitely, we shall have

$$\frac{ds}{dt} = v.$$

This is another equation of the utmost importance. Let us stop for a moment to consider its meaning.

Suppose that the distance of the body from some fixed point in the line of its motion at the instant under consideration is s, and that the whole time that has elapsed since some fixed epoch is t. Had we considered the body at some different time the position would have been different, that is, any change in the value of t is accompanied by some corresponding change in the value of s, or in analytical language, s is a function of t.

Now this being the case, we see that when the particle moves forward through a space δs, its distance from the fixed point be-

comes $s+\delta s$, or is increased by δs; similarly, the time that has elapsed since the fixed epoch is $t+\delta t$, or t has been increased by δt.

Hence δs and δt are the corresponding increments of s and t, and their ratio is $\frac{\delta s}{\delta t}$ and the limit of this ratio, which we have shewn to be equal to v, is the differential coefficient of s with respect to t, consistently with the notation we have used for it. Hence then, when it is known that a body moves so that the distance from a fixed point is always represented by a certain function of the time which has elapsed since a definite instant, the differential coefficient of this function, with respect to t, will express the velocity of the particle at any time. And, conversely, if we know the velocity of a particle at any time as a function of the time, and can find a quantity of which this function is the differential coefficient, this quantity will express the distance of the particle at any time from some fixed point in its line of motion.

In the case of motions arising from natural causes these functions always exist; it would, however, be possible by artificial means to cause motions which should follow no law whatever, or be perfectly irregular; in these cases we should still have v equal to the limit of $\frac{\delta s}{\delta t}$, but since in this case v, and therefore the limit of $\frac{\delta s}{\delta t}$ which is equal to it, would vary from one instant to another arbitrarily, it would be impossible that it should be represented by any function, and consequently could not be the differential coefficient of any function.

Motions of this irregular description are, however, never the subject of mathematical investigation, we may therefore, in all cases that will come under our consideration, adopt the equation $v=\frac{ds}{dt}$, considering $\frac{ds}{dt}$ to be, as the notation indicates, the differential coefficient of s with respect to t.

It must be carefully borne in mind that, though in all these investigations we have spoken of s, v, and t, as space, velocity, and time, we only mean the numerical representations of these quantities expressed on the principles before laid down, and such is always to be understood whenever such language is used, unless the contrary be expressly stated.

8. We will now proceed to describe another kind of motion. We have already considered the case of uniform rectilinear motion, and have seen that the characteristic of this kind of motion is, that at every instant the velocity remains the same; and from the consi-

deration of this kind of motion, we have obtained our measures and numerical representations of velocity. We have also obtained an expression for the velocity in the case where it changes from one instant to another, but we have not at all considered the case of that change taking place according to any law.

Now, conceive a body moving in such a manner that the difference of its velocities at the beginning and end of any interval is equal to the difference of its velocities at the beginning and end of any other equal interval, whatever the duration of the intervals, or at whatever parts of the motion they may be taken. Such motion is called uniformly accelerated motion.

Suppose, for instance, that the velocity at any instant is v_0, that after a time t it is v_1, after another time t it is v_2, and so on.

Then $v_0, v_1, v_2 \ldots$ are the velocities at instants separated by equal intervals t.

Hence then, by the definition of this kind of motion,

$$v_1 - v_0 = v_2 - v_1 = v_3 - v_2, \text{ \&c.},$$

if the velocity be increasing, or,

$$v_0 - v_1 = v_1 - v_2 = v_2 - v_3, \text{ \&c.},$$

if it be diminishing. We may, however, consider the second case as included in the first by the ordinary conventions of negative quantities.

If we add n of these together, we have

$$v_n - v_0 = n(v_1 - v_0),$$

and v_n, v_0 are the velocities at instants separated by an interval nt.

Hence, $\dfrac{v_n - v_0}{nt} = \dfrac{n(v_1 - v_0)}{nt} = \dfrac{v_1 - v_0}{t}$,

and this is true whatever the magnitudes of n and t. We see, then, that in this kind of motion, the ratio of the increments or decrements of the velocity in any intervals of time, is equal to the ratio of those times.

Or, if we consider the numerical representations of the increments of the velocity and of the times in which these increments take place, the ratio of the increment of the velocity to the time in which it takes place is independent of the time.

Let then v be the velocity at a time t from some fixed epoch. And let $v + \delta v$ be the velocity at a time $t + \delta t$ from the same epoch, then we have $\dfrac{\delta v}{\delta t}$ constant for all values of δt, large or small. This is the characteristic property of this kind of motion. Since this ratio

is constant for all values of δt it will be the same when we pass to the limit, and suppose δt indefinitely diminished, in which case the ratio $\frac{\delta v}{\delta t}$ becomes $\frac{dv}{dt}$, the differential coefficient of v with respect to t. This differential coefficient is, however, in this kind of motion, constant. Now conceive two bodies moving each with uniformly accelerated motion and so that the increase of the velocity of one of them in any time is equal to the increase of the velocity of the other in the same time, and consequently so that the ratio $\frac{\delta v}{\delta t}$ and its limit $\frac{dv}{dt}$ is the same for both, then the motions of these two bodies are said to be equally accelerated, or the accelerations of the motions to be equal. Of course it does not follow from this that the velocities are equal at the same time, any more than that two bodies moving with equal velocities are equidistant from the same fixed point or from the fixed points from which their positions are estimated. But whatever difference there is between the velocities at any one time, there will always be the same difference at any other time, because the increase of velocity in the interval is the same for each body. This is the idea we must bear in mind when we say that the motions of two bodies are equally accelerated.

We will now proceed to find a measure for the acceleration of any motion: and first we will explain what is meant when the acceleration of the motion of one body is said to be twice as great as the acceleration of the motion of another.

Here the manner of estimating velocity will supply us with an analogy.

The velocity of one body is twice that of another when the one body passes over in any given time twice the space which the other passes over in the same time.

In a similar manner the acceleration of the motion of one body is said to be twice the acceleration of the motion of another body when the increase of the velocity of the one body in any time is twice the increase of the velocity of the other in the same time.

Thus at any given instant let the velocity of one body be v_0 and of another v_0', and after a time t let the velocity of the one be v_1 and of the other be v_1', then the increments of the velocities of the two bodies in the time t are respectively $v_1 - v_0$ and $v_1' - v_0'$. If $v_1' - v_0'$ be twice $v_1 - v_0$ the motion of the second body is twice as much accelerated as that of the first. This explanation of what is meant, when one motion is said to be twice as much accelerated as another,

must be considered as a definition. It follows at once from the nature of the motion that if this relation holds for any value of t it will hold for all values of t.

Let then δv and $\delta v'$ be the increments of velocity in the same time δt of two bodies moving with uniformly accelerated motion, if the acceleration of the latter is twice that of the former $\delta v' = 2\delta v$ and *vice versa*. Hence we have the equations

$$\frac{\delta v'}{\delta t} = 2\frac{\delta v}{\delta t},$$

$$\text{and } \frac{dv'}{dt} = 2 \cdot \frac{dv}{dt}.$$

And the same reasoning is immediately applicable to any multiple. So that if the acceleration of the motion of one body is n times the acceleration of the motion of another, we have

$$\frac{\delta v'}{\delta t} = n \cdot \frac{\delta v}{\delta t},$$

$$\text{and } \frac{dv'}{dt} = n \cdot \frac{dv}{dt};$$

or, since the ratio remains constant, whatever the magnitude of δt, we need not have it the same on both sides, so that

$$\frac{\delta v'}{\delta t'} = n \cdot \frac{\delta v}{\delta t},$$

$$\text{and } \frac{dv'}{dt'} = n \cdot \frac{dv}{dt}.$$

It now remains to find a numerical representation for the degree of acceleration of the motion of any body. If the acceleration of the motion of any body be considered to be known when the increment of the velocity in any specified time is known, and some particular degree of acceleration be fixed on, then the magnitude of any other acceleration will be determined, when the ratio it bears to that particular degree of acceleration which has been chosen as our standard is determined. Let this particular degree of acceleration be represented by the letter a_1, and let any other acceleration be represented by a, and let $\frac{\delta v_1}{\delta t_1}$ and $\frac{\delta v}{\delta t}$ be the corresponding ratios of the increments of the velocity and time; then if we have

$$\frac{\delta v}{\delta t} = n \cdot \frac{\delta v_1}{\delta t_1},$$

we have also $a = n \cdot a_1$.

From which we have $a = \dfrac{\dfrac{\delta v}{\delta t}}{\dfrac{\delta v_1}{\delta t_1}} \cdot a_1$,

Or, suppressing a which amounts to taking our standard acceleration as our unit, we have the numerical representation of the other acceleration

$$a = \frac{\dfrac{\delta v}{\delta t}}{\dfrac{\delta v_1}{\delta t_1}}.$$

If then we take as our unit that particular degree of acceleration for which $\dfrac{\delta v_1}{\delta t_1} = 1$, we have the equation

$$a = \frac{\delta v}{\delta t} = \frac{dv}{dt}.$$

This unit is always adopted.

The letter f will in future be used to denote the numerical representation of the acceleration, so that our equation is

$$f = \frac{\delta v}{\delta t} = \frac{dv}{dt}.$$

9. All we have said respecting uniformly accelerated motion is equally applicable to uniformly retarded motion, if throughout the whole reasoning we use the decrement of the velocity instead of the increment. In this case we have, if f be the retardation, and δv the decrement of the velocity in time δt,

$$f = \frac{\delta v}{\delta t} \text{ as before.}$$

Or our reasoning will hold if we still take δv to represent the increment of the velocity in time δt, and consider it negative when the velocity is diminishing or the motion is retarded: that is, essentially negative, not, having a negative sign prefixed. This is merely the ordinary algebraical generalization.

In this case our equation

$$f = \frac{\delta v}{\delta t} = \frac{dv}{dt},$$

gives for f a negative value when the motion is retarded, agreeably to the opposition expressed by the signs + and −.

If then in the result of any investigation the expression for the acceleration should be negative, we must conclude from that circumstance that the motion is retarded, and the magnitude of the expression for the acceleration, independently of its sign, will give the magnitude of the retardation.

We have dwelt at some length on this particular kind of motion, because, as will be seen afterwards, it is of great importance.

10. In considering the motion of a body moving with variable velocity, we saw at once that at every instant the body was moving with some definite velocity, and it only remained to express that velocity in terms of the space described and the time of describing it: which was easily done.

It is not, however, so easy to see that when a body moves in any manner whatever (excepting that sort of motion which we have termed irregular) there is, at every instant, some definite degree of acceleration of the motion; that is, to conceive some uniformly accelerated motion in which the velocity is being increased at the same rate as in the proposed motion at the instant under consideration.

That this is the case will nevertheless be seen by a little careful consideration of the nature of acceleration. It may perhaps be rendered clearer by the following analogy.

Velocity is the rate at which the space described by the body increases; the acceleration is the rate at which the velocity acquired by the body increases. When the space described increases uniformly, the motion is said to be uniform; when the velocity increases uniformly, the motion is said to be uniformly accelerated. Carrying the analogy one step farther, when the motion is variable, the velocity at any instant is the rate at which the space described is increasing at that instant; when the motion is not uniformly accelerated, the acceleration at any instant is the rate at which the velocity is increasing at that instant.

This must not be considered as a proof, since, in fact, there is nothing to be proved, but merely as an explanation tending to facilitate the formation of the idea.

Let then f be the acceleration at some instant at which the velocity is v, and after an interval δt, let $f + \delta f$ be the acceleration, and $v + \delta v$ the velocity.

Then $\frac{\delta v}{\delta t}$ will lie between f and $f + \delta f$.

But when δt is indefinitely diminished, δf is also indefinitely diminished, and ultimately vanishes, and we have therefore,

$$\text{limit of } \frac{\delta v}{\delta t} = \frac{dv}{dt} = f.$$

When the motion is uniformly accelerated, it is indifferent whether we take $\frac{\delta v}{\delta t}$, or the limit of that quantity $\frac{dv}{dt}$, since the quantity being constant is equal to its limit. When, however, the acceleration is not uniform, the ratio $\frac{\delta v}{\delta t}$ is no longer independent of the magnitude of δt, and we must take as our numerical representation of the acceleration $\frac{dv}{dt}$, the limit of the ratio $\frac{\delta v}{\delta t}$.

Hence we have generally,

$$f = \frac{dv}{dt}.$$

Also, in uniformly accelerated motion,

$$f = \frac{\delta v}{\delta t}.$$

11. We have then in uniform motion,

$$v = \frac{s}{t};$$

also we have, whether the motion be uniform or not,

$$v = \frac{ds}{dt};$$

and whether the motion be uniformly accelerated or not,

$$f = \frac{dv}{dt}.$$

From this last equation we obtain at once,

$$f = \frac{d^2s}{dt^2}.$$

When the motion is uniformly accelerated, f, and therefore $\frac{dv}{dt}$ and $\frac{d^2s}{dt^2}$ will be constant. When the motion is not uniformly accelerated, f will vary from one instant to another, and will therefore be a function of t, the time which has elapsed since some fixed epoch. It does not always happen that $\frac{dv}{dt}$ and $\frac{d^2s}{dt^2}$ are expressed explicitly as functions of t; but they must be either functions of t, or of some quantity which is itself a function of t.

When we know the expressions for these quantities, and are able to integrate them, we can find the values of the velocity and of the space described at every instant throughout the whole duration of the motion.

12. Hitherto we have only considered rectilinear motion; we will now proceed to discuss the method of representing analytically motion which is not rectilinear. When the term Velocity was first used, no definition was given of it, and it will appear from a little consideration, that the idea employed is equally applicable whether the motion is in a straight line or a curved one. We can conceive a point moving along a curved line with different degrees of quickness, and our idea of its pace or velocity is as definite as if its path were a straight line. And if s is its distance at time t from a fixed point in its path, measured along that path, $\frac{ds}{dt}$ is its velocity at that time, and $\frac{d^2s}{dt^2}$ is the measure of the acceleration of its motion. In this case, however, there is something besides the velocity required to complete our knowledge of the state of motion of the body: we must know the direction of its motion at every instant. Now to determine the position in space of any point we use three co-ordinates, that is, the distances from some fixed point O of its projections on three fixed lines passing through O. Call the three lines Ox, Oy, Oz, and let L, M, N be the projections of the point P on them made by planes parallel respectively to yOz, zOx, xOy. Then when OL, OM, ON are known, the position of P is completely determined; and when the velocities of L, M, N along Ox, Oy, Oz, that is, the rates of increase of OL, OM, ON are known, the velocity and direction of P's motion can be found from the geometry.

In the simple case where Ox, Oy, Oz are mutually at right angles to each other this is easily expressed.

Calling the coordinates of P, x, y, z, the velocities of L, M, and N will be

$$\frac{dx}{dt}, \quad \frac{dy}{dt} \text{ and } \frac{dz}{dt},$$

and if v be the velocity of P, α, β, γ the angles which its direction makes with Ox, Oy, Oz,

$$v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2},$$

$$\cos\alpha = \frac{1}{v} \cdot \frac{dx}{dt}, \quad \cos\beta = \frac{1}{v} \cdot \frac{dy}{dt}, \quad \cos\gamma = \frac{1}{v} \cdot \frac{dz}{dt};$$

$\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$, are called the resolved parts of the velocity of P in the directions Ox, Oy, Oz.

When Ox, Oy, Oz are not at right angles to each other, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ are called the components in those directions. Hence it appears that the component of a velocity in any direction depends on the directions in which the other components are taken; when, however, the resolved part in any direction is spoken of, the other components are understood to be at right angles to it.

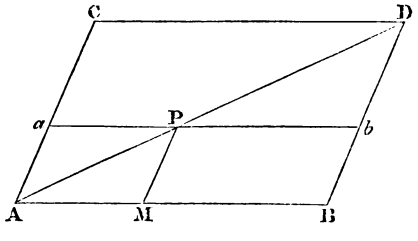
13. The propriety of the term components will be better seen from the following proposition called the parallelogram of velocities. This proposition may be stated as follows:

If a point be moving in any direction with any velocity, and if to this motion another motion be superadded in any direction, then if the two simple velocities be represented in magnitude and direction by lines drawn from any point, the actual velocity will be represented in magnitude and direction by the diagonal of the parallelogram described on these two lines.

What is meant by superadding a motion will be best understood by an illustration. Suppose a point to be moving uniformly across a table in a straight line. This will be the first motion. Now while this motion relatively to the table continues, suppose the table itself to be moved uniformly in some other direction, this will be the second motion superimposed on the particle: neither the one nor the other will be the motion of the particle relatively to the floor of the room.

It must not be supposed that we here assert that if a ball were rolling or sliding across a table, and the table were to be pushed forward, the motion of the ball relatively to the table would be unaffected; whether this would or would not be the case, is a subject for after-investigation: all we suppose is, that a point, a merely geometrical point if you please, is constrained artificially to advance uniformly in a straight line relatively to the table.

Let AB be a line chosen arbitrarily to represent in magnitude and direction the first velocity of the point, and let t be the time in which the particle would describe AB with this velocity.



Now, in order to impress on it the second velocity, suppose it inclosed in a tube which originally coincides with AB. Then from the first motion the point would move uniformly from the end A of the tube to the end B in the time t, and any velocity in any direction would be superimposed on the former velocity by moving the tube parallel to itself in that direction and with that velocity, the other motion relatively to the tube being supposed to continue unaffected by the motion of the tube.

Let the velocity of the tube be such as would carry it uniformly from the position AB to the position CD in the time t. Then AC will represent in magnitude and direction the second velocity superimposed on the former: it is to be shewn that the diagonal AD will represent the actual velocity of the particle resulting from these two.

At the end of the time t, the point will be at D; at the end of any shorter time t_1, let ab be the position of the tube: then, since its motion is uniform, we have

$$\begin{aligned} t_1 : t &= Aa : AC \\ &= aP : CD \text{ by similar triangles,} \\ &= aP : ab. \end{aligned}$$

therefore, since the motion of the point in the tube is uniform, P is the position of the point, that is, the point at every instant is in the diagonal and therefore moves along it. Also

$$\begin{aligned} AP : AD &= Aa : AC, \\ &= t_1 : t, \end{aligned}$$

which shews that the point moves uniformly along the diagonal in the time t. Therefore AD represents the resultant velocity in magnitude and direction. Which was to be shewn.

From this it follows that if a particle be moving in any direction, we may suppose the velocity to be the resultant of two velocities in any two directions in the same plane with the actual velocity, or, if we extend the parallelogram of velocities to a parallelepiped, of three velocities in any three directions.

Since the sum of the squares of three adjacent sides of a right solid is the square of the diagonal, if v_x, v_y, v_z represent three velocities in directions at right angles to each other coexisting in a body, v the actual velocity,

$$v^2 = v_x^2 + v_y^2 + v_z^2,$$

and if α, β, γ be the angles between the direction of v and those of v_x, v_y, and v_z,

$$v_x = v \cos \alpha, \quad v_y = v \cos \beta, \quad v_z = v \cos \gamma.$$

It must not be supposed that the parallelogram of velocities and what follows are in reality different from the method in Art. 12. It has, however, afforded an opportunity for explaining the sense in which we shall afterwards speak of adding one velocity to another, and of any number of velocities coexisting.

14. If any number of velocities be superadded at the same time to a body (in the sense which we have just explained) the resultant velocity may be easily found.

We may take the resultant of any two, and by combining this resultant with another, obtain the resultant of three, and so on to any number.

Or we may proceed as follows:

Fixing arbitrarily on three directions, we may resolve each velocity into its components in these three directions, and add together the components in each of the three directions. We shall then have the resultant velocity expressed in terms of three components, and can find its magnitude and direction at once by combining these three.

15. Having explained certain terms which will constantly occur, and described certain kinds of motion which will continually come under our notice, we will now proceed to investigate the causes which would produce those motions, and to establish rules for determining the nature of the motions which would result from the action of specified causes.

We have hitherto used the terms point, particle, body, merely to denote a position in space. A geometrical point supposed capable of motion would have answered our purpose equally well. We will now, however, confine our attention to such bodies or points as we have described in Art. 1. And this is to be understood whatever term is used to express it.

Force is defined to be "any cause which produces, or tends to produce, any change in a body's state of rest or motion."

In the science of Statics, however, we are only concerned with forces which tend to produce a change in a body's state of rest. In the present subject the whole of the definition is required.

The first inquiry that suggests itself is, What will become of a body if left to itself, that is, if not influenced by any bodies external to itself?

Suppose a body in motion; we are unable to say, *à priori*, whether if left to itself the motion will gradually die away and the body ultimately stop; whether it will move in a straight line or be

deflected from it in any direction; whether or not it will have a tendency to move in a circle; in fact, we cannot say any thing about it. It is found, however, that it will move in obedience to the following law, which is called the First Law of Motion.

A body in motion not acted on by any external force will move in a straight line with uniform velocity.

This law assures us of two facts.

First, that matter has no property inherent in itself which enables it to change the direction of its motion, or to alter or destroy its velocity; and

Secondly, that the molecular forces which keep together the ultimate atoms of which our particle is composed have no such tendency.

It follows from the definition of force, that if the body be not acted on by any force, no change can take place in its motion, and therefore it must move on uniformly in a straight line. For any change must have some cause, and by the definition, the name "force" is given to that cause, whatever it be.

The First Law of Motion, then, assures us that when there is no external force acting on the body, there is no force acting at all which can avail to change the motion.

We will, at present, say nothing about the proof of this law. The true nature of the proof, and the true bearing of the experiments which suggested the law, will be better appreciated when farther progress has been made in the subject.

16. Before proceeding farther, we will make a remark which must never be lost sight of. It is this. Force, by whatever means exerted, or from whatever cause arising, is always necessarily essentially of the same nature.

Forces may differ from one another in their intensity, and in their directions, and in the points at which they are exerted, but they are always forces, and as such are always magnitudes of the same kind.

Thus, a force may be exerted by means of a string, in which case it is called tension. It may be exerted by pushing one body against another, in which case it is called pressure or reaction. We may have the force of friction, pressure exerted by a spring, by a body's weight which arises from the attraction of the earth, by any other attraction or repulsion whatever, but still all are the same.

Forces may produce equilibrium, or they may produce motion, but still the force itself is the same.

When, therefore, we use the term "force," we always mean such a force as a pressure or tension.

Two forces are equal to one another when a point on which they act in opposite directions continues at rest under their action.

Two equal forces acting on the same point in the same direction, constitute together a force double of either of them.

From these two definitions we are able to represent a force numerically, by referring it to some standard force as a unit.

17. We will now consider the effect of a force when acting on a material point.

Suppose the point at rest originally; when the force acts it will generate a velocity in the particle, and if the force acts for a certain time and then ceases, we know from the first law of motion that the particle will continue to move in a straight line with the velocity which it had when the force ceased acting.

Now suppose a force exactly equal to this one to act for exactly the same time on another particle originally at rest, exactly similar and equal to the other one; by the end of the time this particle will have acquired the same velocity as the other, and will continue to move on with this same velocity when the force ceases acting.

Or, in other words, when a given force acts on a given particle, originally at rest, for a given time, there is always some definite velocity which it will generate in the time.

Now suppose that a force acts for a certain time on a particle and produces a certain velocity, if it goes on acting for another equal time in the same direction, it will increase the velocity, but we cannot, *à priori*, say by how much. Here then we are stopped again.

Also, suppose that together with our force another equal force acts on the body for the time specified, in the same direction with it, thus amounting to a double force, we cannot say, *à priori*, what proportion the velocity generated in any given time, by these two forces, will bear to that which would have been generated in the same time by one only.

To meet these two difficulties we have another rule to guide us, called The Second Law of Motion, which may be thus stated.

When any number of forces act upon a particle in motion, the effect of each force on the velocity of the particle is the same in magnitude and direction as if it acted singly on the particle at rest.

The best commentary on this law will be the manner in which it is applied. When a force is designated by a letter (as P) that letter must be considered as representing numerically the intensity

of the force, estimated by reference to some standard unit. All forces whose intensities are equal and which act in the same direction will be considered as the same force, from whatever different causes they may arise. For instance, a body pulled by a string, and another attracted by a magnet in the same direction and with the same intensity, would be considered as acted on by the same force. Forces of different intensities, even though they arise from the same cause (as from the attraction of the same body at different distances), are considered different.

18. Suppose then a force P to act on a particle for a time t in which it generates a velocity v; then if it acts during another interval t it will, by our law, produce the same effect as during the first interval, and therefore during this second interval t it will generate an additional velocity v, so that after a time $2t$ the particle will be moving with a velocity $2v$, and generally after a time nt it will be moving with a velocity nv, which may be expressed in words by saying that when a constant force acts upon a particle, the velocities generated from rest in different times are to one another in the same ratio as the times in which they are generated.

From the manner in which the velocity of the particle increases, we see that its motion will be of that kind which we have called uniformly-accelerated motion.

Hence then we have the conclusion, that a constant force acting on a particle will produce uniformly-accelerated motion. And this result will of course be true for every constant force, and for every particle. We will however at present confine our attention to the same particle or to particles which are alike in every respect.

19. Now suppose two forces each, P, to act together in the same direction on the same particle for the time t. Each would in this time, if it acted separately, generate from rest the velocity v. But by our law the effect of each force will be produced as if the other did not act. Each therefore will generate its velocity v; so that the whole velocity generated in the time t will be $2v$: and these two forces P constitute a force $2P$. Hence then a force $2P$ generates in any time twice the velocity which a force P would generate: and by similar reasoning it appears that a force nP generates n times the velocity which a force P would generate in the same time. And from what has preceded we know that the motion of the particle under the action of this force nP will be uniformly accelerated: and the increment of the velocity in any time will be n times the in-

crement in the same time under the action of the force P: or, in other words, the acceleration of the motion under the action of the force nP is n times the acceleration under the action of the force P. Hence then, if P and P' be any two forces, and f and f' the accelerations of the motions which they produce in the same particle, we shall have

$$\frac{P}{P'} = \frac{f}{f'}.$$

Here then we have come to the following conclusion. When a constant force acts on a particle it produces uniformly-accelerated motion during its action. And the accelerations of the motions which different forces produce in the same or equal particles are to each other in the same ratio as the forces which produce them.

20. We have hitherto considered the action of forces on particles exactly similar and equal in all respects, and, consequently, with exactly the same properties. It remains to consider the action of forces on different particles, and to do this we must first endeavour to convey an idea of what is meant by the term "mass."

The definition generally given is that the "mass" of a body is the quantity of matter in it. Unless, however, it be accurately stated how this quantity of matter is to be estimated, we are still as far off as ever from knowing what is meant by the term "mass."

We will take the above definition, and to render it definite will join it with the following.

The masses of two particles are said to be equal when the accelerations of the motions generated in them by the action of equal forces are equal. This may or may not be the same thing as saying that their volumes are equal or that their weights are equal. That must be determined by after considerations. We here give the definition that will be of most service to us, being perfectly at liberty to give whatever arbitrary definitions we please, on the sole condition of adhering to them throughout.

If we join together two particles whose masses are equal, we shall have a particle whose mass is double that of either of the constituent particles. This is our definition of a double mass, and so on for any multiple. And exactly as was the case with respect to space and time we shall have the numerical representation M of any mass, that number which expresses the ratio which its mass bears to that particular mass which we have chosen as our unit.

21. Now, suppose a force P acting on a mass M to produce a motion whose acceleration is f, and suppose n such particles to be arranged side by side close together, and each to be acted on by a

force P in the same direction. Then it follows from our definition of equal masses that the acceleration of the motion of each will be f, and therefore since they start together they will move on together as one body.

It will not affect this motion if we suppose all these particles united so as to form one particle, in which case we have a motion whose acceleration is f produced by the action of a force nP on a mass nM.

Now a motion whose acceleration is f is produced by the action of a force P on a mass M, and therefore from what has been proved before, a motion whose acceleration is nf would be produced by the action of a force nP on a mass M.

And this is true whatever be the magnitude of n.

Hence the accelerations of the motions produced in different particles by the action of the same force nP are inversely proportional to the masses of those particles.

Or, generally, if f and f' be the accelerations of the motions produced in masses M and M' by forces P and P', and f_1 the acceleration produced in the mass M' by the force P, we have

$$\frac{f}{f_1} = \frac{M'}{M}.$$

And it was shewn in Art. 19 that

$$\frac{f_1}{f'} = \frac{P}{P'},$$

$$\text{whence it follows that } \frac{f}{f'} = \frac{P}{M} \cdot \frac{M'}{P'},$$

$$\text{or } f : f' = \frac{P}{M} : \frac{P'}{M'}.$$

22. It has been shewn in the last few articles that when a constant force P acts on a particle whose mass is M, it produces uniformly-accelerated motion, and that the amount of the acceleration varies as the fraction $\frac{P}{M}$. If then two forces P and P' act on two particles whose masses M and M' are such that $\frac{P}{M} = \frac{P'}{M'}$, the motions of these two particles will be equally accelerated, or the power which the force P has of accelerating the motion of M is equal to the power which the force P' has of accelerating the motion of M'. And if $\frac{P}{M} = n\frac{P'}{M'}$, the power which P has of accelerating the motion of M is n times as great as the power which P' has of accelerating the motion of M' This power which a force has of accelerating

the motion of a particle is called the accelerating-force of that force on the particle; and in the first of the two cases above we say that the accelerating-force of P on M is equal to the accelerating-force of P' on M', and in the second the accelerating-force of P on M is n times the accelerating-force of P' on M'. From this definition it appears that the accelerating-force of any force on any particle is proportional to the acceleration of the motion which that force, acting alone, would produce in the particle, and for convenience the unit of accelerating-force is so chosen that the same number represents the two.

It may seem at first sight to have been unnecessary to make this distinction between the accelerating-force of a force and the acceleration of the motion produced by the force; but a little consideration will shew that the idea of acceleration can be formed without any reference to force and mass, while the accelerating-force on a definite mass is a property of the force which exists even though the effect of the force may be counteracted or modified by other causes.

The term "accelerating-force" will be constantly used as an abbreviation of the longer expression "the accelerating-force of a force on a particle;" it must not however be concluded from this independent use of the expression, that accelerating-force is itself a force of the kind referred to in Art. 16. The term is a bad one, inasmuch as it favours this misconstruction, but will lead to no error if the preceding explanation is borne in mind, and the term considered as one word, equivalent to some such term as acceleretiveness.

23. It has been shewn that the acceleration f of the motion produced in a mass M by a force P is proportional to $\frac{P}{M}$. To simplify our expression, such a relation is assigned between the units of force and of mass that

$$f = \frac{P}{M}.$$

If we consider the unit of mass as the arbitrary quantity in this case, it amounts to defining the unit of force as that force which will produce the unit of acceleration in the unit of mass, or the force whose accelerating-force on the unit of mass is unity.

Combining the preceding equation with those obtained before, we have

$$v = \frac{ds}{dt},$$

$$f = \frac{P}{M} = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

24. We have hitherto confined our attention to the action of a constant force. The force P which we have supposed to act did not vary from one instant to another, and we have shewn from the second law of motion, that the motion produced in a particle by such a force was uniformly accelerated. We will now consider the action of a variable force, that is, a force whose intensity is continually changing.

Suppose, for the sake of definiteness, that the force is increasing; and at a time t from some fixed epoch let its intensity be P, and at a time $t+\delta t$ suppose it to be $P+\delta P$.

Let M be the mass of the particle on which it is acting, and v the velocity with which it is moving at the time t, $v+\delta v$ the velocity with which it is moving at the time $t+\delta t$, and $v+\delta v_1$ and $v+\delta v_2$ the velocities with which it would have been moving at the time $t+\delta t$, had the force remained constant during the time δt, with the respective intensities P and $P+\delta P$. Then the velocity $v+\delta v$ will lie between $v+\delta v_1$ and $v+\delta v_2$, or δv will lie between δv_1 and δv_2 and therefore $\frac{\delta v}{\delta t}$ will lie between $\frac{\delta v_1}{\delta t}$ and $\frac{\delta v_2}{\delta t}$.

Now from our investigations with reference to a constant force we know that

$$\frac{\delta v_1}{\delta t} = \frac{P}{M},$$

$$\text{and } \frac{\delta v_2}{\delta t} = \frac{P+\delta P}{M}.$$

Hence then $\frac{\delta v}{\delta t}$ always lies between $\frac{P}{M}$ and $\frac{P}{M}+\frac{\delta P}{M}$.

But as δt decreases δP decreases, since the intensity of the force is supposed to vary continuously: and in the limit $\frac{\delta v}{\delta t}$ will become $\frac{dv}{dt}$, the two quantities between which its magnitude always lies will coincide, and it must therefore be equal to either of them; we have therefore the equation

$$\frac{P}{M} = \frac{dv}{dt}.$$

$$\text{and, consequently, } \frac{P}{M} = \frac{d^2s}{dt^2}.$$

This equation is true at every instant during the motion, P representing the magnitude of the force at that instant. If the letter P be considered as standing for such a function of t, or quantities depending on t, as will express the magnitude of the force at every proposed instant, the equations

$$\frac{P}{M}=\frac{d^2s}{dt^2} \text{ and } v=\frac{ds}{dt}$$

when solved will give the velocity of the body and the space it has moved over, in terms of the time elapsed since a fixed epoch.

When forces arising from different causes act on a particle in the direction of its motion, P in the preceding equation must be considered to represent the resultant of all such forces; or the equation might be written

$$M\frac{d^2s}{dt^2}=M\frac{dv}{dt}=P_1+P_2+\&c.$$

P_1, P_2, &c. being essentially positive or negative as they tend respectively to increase or diminish s.

25. Suppose a body whose mass is M to be moving with a velocity v, then the product of the numbers M and v, viz. Mv, is called the momentum of the body.

The assignment of this name to this particular product may be considered perfectly arbitrary. As, however, the product presents itself in a very extensive class of problems, as will be seen hereafter, it is found convenient to have a name expressing it. If we put the equation

$$\frac{P}{M}=f$$

under the form $Mf=P$,

we see that the product of the numerical representatives of the mass and the accelerating-force of the force P on it, is the numerical representative of the force P. The name of moving-force is given to this product. The terms, force, moving-force, and pressure, are however used indiscriminately.

Since $f=\frac{dv}{dt}$ and M is constant,

$$Mf=M.\frac{dv}{dt}=\frac{d.Mv}{dt}.$$

Hence, the moving-force Mf, bears exactly the same relation to the momentum Mv, that the accelerating-force of the force P, viz. f, does to the velocity v.

This, however, is not of much importance.

26. When a pressure P acts on a body whose mass is M, and generates motion in it, if v be the velocity at time t,

$$P=\frac{d.Mv}{dt},$$

where P may be either constant or variable, and, when the latter is the case, must be either explicitly or implicitly a function of t.

Integrating this equation, we have

$$Mv = \int P dt:$$

at the times t_1 and t_2, let v_1 and v_2 be the velocities of the body, then

$$Mv_2 - Mv_1 = \int_{t_1}^{t_2} P dt,$$

or the definite integral $\int_{t_1}^{t_2} P dt$ expresses the change in the momentum of the particle which takes place between the times t_2 and t_1 from the action of the force P.

When P is known as a function of t, this integration can be effected, and the change of momentum deduced from it.

It sometimes happens that, although P is not known as a function of t, and although the interval $t_2 - t_1$ is not known, the value of the definite integral is known. Or, in other words, we sometimes know the whole change in velocity or momentum which is caused by a certain force, when we do not know the law of the force, or the time during which it acts. When this is the case, although from the velocity of the particle before the action of the force commenced, we can determine the velocity when it ceased to act, or the converse, we cannot determine the position of the body, because we are ignorant of the nature of its motion during the time $t_2 - t_1$.

In the cases, however, of this kind which come under our notice, the interval $t_2 - t_1$ is extremely short, so that we may neglect without sensible error the change of position of the body which takes place during that interval, or consider it in the same position at the time t_2 as it was at the time t_1.

The force P, on the contrary, is generally so great as to produce a considerable change in the velocity v, or the momentum Mv.

In these cases the change of velocity from v^1 to v_2 is considered to take place instantaneously: the action of the force is said to be impulsive, and the definite integral $\int_{t_1}^{t_2} P dt$ is called the "impulse," or "blow," perhaps an abbreviation for "effect of the impulse," or "measure of the impulse."

When we consider the definite integral $\int_{t_1}^{t_2} P dt$, or the change of momentum $M(v_2 - v_1)$, as expressing the magnitude of the impulse, we fix on the particular magnitude of the impulse which we take as our

unit, namely, that impulse which if applied to a unit of mass at rest would cause it to move with the unit of velocity. Thus, if we denote the impulse by R, and suppose the body originally at rest, we have

$$R = Mv,$$

and R will equal unity when

$$M = 1, \text{ and } v = 1.$$

The term "impulse" must not be confounded with the term "impulsive-force."

The impulsive-force is the force P, which is generally very large and not known, which acting through the finite but extremely short time $t_2 - t_1$, produces a finite effect.

The impulse is the whole action of the impulsive force during the interval $t_2 - t_1$, and is, as we have seen, generally known.

The impulsive force P is of the same nature as all other forces, and is, of course, comparable with them in magnitude.

The impulse R is entirely different in its nature from forces, and can only be compared in magnitude with other impulses.

The magnitude of R depends jointly on P and $t_2 - t_1$, and though the latter is very small, yet its value in one case may bear any ratio to its value in another, without violating this condition.

Since an impulsive-force is not different from any other force, all the results that we shall arrive at for forces generally will be true of impulsive forces, unless there is some particular consideration which excludes them.

It would be premature at present to describe the nature of the action which takes place between bodies when an impulse is produced, or, as it is said, when there is "impact" or "collision." All this will be fully discussed hereafter. The class of problems to which it leads are particularly beautiful, and have this advantage, that all the differential equations to which they lead are of the first order instead of the second.

27. From the identity of our ideas of force, whether considered as in equilibrium or producing motion, it follows that when any number of forces arising from different causes act at the same time on a particle in motion, we may replace these forces by their resultant calculated on the principles of statics, and if we suppose this force to change from instant to instant, so as always to be the resultant of the variable forces which act on the particle, the effect of this single force will be the same as the effects of those forces of which it is the resultant. Or, which is more convenient, we may

replace the forces by the three components of that resultant in three constant directions.

Let u, v, w be the components at time t of the velocity of the particle in the direction of three axes; X, Y, Z the sums of the components in directions of the axes of all the forces which act on the particle at time t.

$X+\delta X$, $Y+\delta Y$, $Z+\delta Z$ the same quantities at time $t+\delta t$. δX, δY, δZ may depend on δt explicitly, and also on the change of position and velocity of the particle, but in all cases they will vanish with δt.

Now we are told by the second law of motion, that each of these forces will produce the same effect on the velocity of the particle during the time δt as if the particle were at rest at the commencement of δt, and that force were the only one acting.

Hence the variable force X will produce in that time a velocity δu which lies between $\frac{X\delta t}{M}$ and $\frac{(X+\delta X)\delta t}{M}$, and the other forces will produce effects in their own directions, namely,

$$\delta v \text{ between } \frac{Y\delta t}{M} \text{ and } \frac{(Y+\delta Y)\delta t}{M},$$

$$\text{and } \delta w \text{ between } \frac{Z\delta t}{M} \text{ and } \frac{(Z+\delta Z)\delta t}{M}.$$

so that the components of the velocity at the time $t+\delta t$ will be $u+\delta u$, $v+dv$, $w+\delta w$, where δu, δv, δw are properly the increments of u, v, and w.

Hence, passing to the limit, we have

$$M\frac{du}{dt}=X,\ M\frac{dv}{dt}=Y, \qquad \text{and } M\frac{dw}{dt}=Z.$$

If x, y, z are the coordinates of the particle at time t,

$$u=\frac{dx}{dt},\ v=\frac{dy}{dt},\ w=\frac{dz}{dt},$$

$$\text{and } M\frac{d^2x}{dt^2}=X,\ M\frac{d^2y}{dt^2}=Y,\ M\frac{d^2z}{dt^2}=Z.$$

These equations must be considered as the fundamental equations of dynamics where the motion of a particle only is considered.

28. It may be objected to what has preceded, that we have used terms which have in themselves a meaning in ordinary language, and that we have assigned to these terms a sense differing

from, or only partially agreeing with, their ordinary meaning, and that this sense has been assigned arbitrarily.

To this we reply that we are perfectly at liberty to give any arbitrary definitions that we please, provided only that we adhere to them throughout the subject.

The Science of Mechanics, however, is particularly unfortunate in this respect, that the terms used in it have, in addition to the meaning given them by definition, a meaning in common conversation. It arises from the fact, that in the early ages of the science erroneous, very erroneous notions were entertained of the principles of the science, and the phraseology adopted expressed those erroneous notions. When the true principles of the science were gradually developed, it was found convenient to retain the old names expressing the same analytical combinations and numerical quantities as before, though all idea of the name expressing any principle was given up.

In popular language, however, the old notions still cling to the words.

Such terms are "momentum," "accelerating-force," "vis viva," &c.

We caution the student, once for all, to take the words in the sense assigned them by definition and in no other.

29. It is a matter of every-day experience, that bodies of any material if left unsupported fall to the ground, and that some amount of force is requisite to counteract this tendency. This is caused by the attraction of the earth on the body. This force, with which the earth pulls the body, is called the weight of that body, and it is found by experiment that the weight of a given volume of any particular substance (as lead) is very nearly the same at all points on the surface of the earth, and for small distances above it. Assuming, for the present, that it is accurately so, a heavy particle when abandoned to itself *in vacuo* is acted on only by this constant force, and will consequently, as has been shewn in the preceding articles, move with uniformly-accelerated motion. It must not be imagined that we assert here that we have proved that a particle abandoned to itself *in vacuo* will move with uniformly-accelerated motion, but we have proved it on the assumption of the two laws of motion and of the fact that the weight of a body at the same place is constant.

If we take two similar and equal particles, these will have the same weight and the same mass, and consequently the accelerating forces of their weights will be the same, and if the particles

are let loose together they will move on together: and this will not be altered if we suppose them joined, and so for any number. That is, for the same substance the accelerating force of the attraction of the earth is constant.

This might have been deduced at once from the consideration that both the masses and weights of different portions of the same substance would be proportional to the volume and therefore to each other.

It is found by experiment that, for bodies of all substances, and not for bodies of the same substance only, the accelerating-force of the earth's attraction is the same, and such as to generate in any body whatever, *in vacuo*, in a second, a velocity with which it would move over 32·2 feet in a second. If then we take a second as the unit of time and a foot as the unit of space, the numerical representation of the accelerating-force of the earth's attraction will be 32·2. This is written g, and is called the accelerating-force of gravity.

If then W be the weight of a body and M its mass, we have the equation $W = M.g$

By the weight we mean the force with which it presses downwards or which is requisite to support it.

Since the quantity g is the same for all bodies, we see that the weight and mass of a body are proportional. The quantity g, though very nearly constant, is not accurately so. Its magnitude is different in different latitudes, and at different altitudes above the earth's surface. The variations, however, for positions within our reach are so small that they are generally omitted, and g is considered constant.

When we are concerned with very great heights on the earth, or with bodies at distances from the earth, we are obliged to take into account the variations in its magnitude. The law of its variation, and the great principle of which it forms a part, will all be fully explained in their places. They have however more to do with the facts than with the principles of dynamics.

30. The fact that the accelerating-force of the attraction of the earth on any body is the same whatever the weight of the body, has led in the first class of problems that we are concerned with, viz. falling bodies, to the neglect of the weight and mass of the bodies and the consideration only of the accelerating force of that weight, viz. g. And this has extended itself very much to other problems, so that we constantly meet with the term accelerating-force applied absolutely, without any reference to the force or mass on which it acts. If in all such cases we understand by the term "accelerating-

force" such an expression as "a force whose accelerating-force" it can create no ambiguity, and is a very admissible abbreviation. For instance, we should find it said, "A body descends under the action of a constant accelerating-force g" Instead of "A body descends under the action of a force, whose accelerating-force is constant and equal g" The term "accelerating-force" is in constant use in this sense, and, provided this meaning is always attached to the word, there is no objection to its use. In Dynamics the weight of a body is scarcely ever used, we employ instead $M.g$, because it very often happens that the factor M may be divided out from both sides of our equations.

31. The unit of force has been chosen so that the equation $P = Mf$ may hold. In common affairs the unit of force is chosen arbitrarily, and this unit is generally used in the science of Statics. It is necessary, therefore, to explain how the relation between the two may be found. For this purpose, we must be able to express the weight of the unit of mass in terms of the common standard; in pounds, for instance.

Let then the unit of mass weigh n lbs. and let W be its weight in terms of the dynamical unit; from the equation $P = Mf$ we have

$$W = Mg = g,$$

and therefore g and n express the same weight referred to different units. Hence, to reduce any dynamical force P to lbs. we must multiply it by the factor $\frac{n}{g}$.

32. We will now proceed to explain the theory of gravitation which was mentioned above, and the nature of the reasoning on which the truth of it, and of the laws of motion, depends.

When one particle of matter is said to attract another, it is meant that if these particles be placed at a distance from each other and then abandoned to themselves, they will commence moving towards each other, or, that if they be not so abandoned to themselves, a certain degree of force must be applied to them to prevent their approaching each other. The intensity of this force for different particles at different distances depends on the law of attraction and its intensity.

For instance, every substance is attracted to the earth, and we see that if any substance is left to itself it immediately commences moving towards the earth, or if not, a certain degree of force is required to support it.

The law of gravitation is this.

Every particle of matter attracts every other particle of matter with a force which varies directly as the product of the masses of the two particles, and inversely as the square of the distance between them.

Thus if m and m' be the masses of two particles, and r the distance between them, the force with which they attract each other, or which it would be necessary to apply to each of them to keep them asunder, will be $C \cdot \frac{m \cdot m'}{r^2}$, where C is some constant quantity which is independent of the masses of the particles and of the distance between them.

Since the force acting on m' to move it towards m is $C \cdot \frac{m \cdot m'}{r^2}$, the accelerating-force of this force on m' is $C \cdot \frac{m}{r^2}$, which is independent of the mass of the attracted particle.

That mass is frequently chosen for the unit of mass whose attraction at the unit of distance has the unit of accelerating-force; in this case $C = 1$, and the attraction between two particles is expressed by $\frac{m \cdot m'}{r^2}$.

The determination of this mass, that is, the expression of it in terms of a given volume of some specified substance, requires most difficult and delicate experiments. It has, however, been effected with great accuracy. An account of the experiments will be given hereafter.

33. Since, then, every particle of matter in the universe attracts every other particle of matter according to the law just explained, we are able by this law and the equations of motion which we have obtained, to calculate to any degree of accuracy the motions of any one of the heavenly bodies, and to predict the exact direction in which it will be seen from the earth at any proposed future instant, however remote.

These calculations are very intricate and laborious, and involve many complicated considerations. Astronomical Instruments have also arrived at extreme perfection, so that we are able also to observe at any instant the directions in which these heavenly bodies are seen from the earth. And the more exact our calculations are made, and the more perfect our instruments and methods of observing, the more minute are found to be the agreements between these results of calculation and observation.

Nothing can afford a more complete proof of the truth of our assumptions than this. In order, however, that the proof may be fully appreciated, the whole of the processes of calculation and observation should be familiarly known. It is impossible, without some acquaintance with an observatory, to realize the degree of accuracy to which observations can be carried, and it is this accuracy which conveys the most perfect conviction to the mind.

Perhaps the recent discovery of the planet Neptune may afford a more *striking* proof of the truth of these laws. Some minute disagreements between the results of calculation and observation had long been noticed in the positions of the planet Uranus; and it had been suspected that there was some other body in our system, to the neglect of whose action these differences might be due. But it was long deemed past the powers of calculation to discover where a body must be to produce exactly such differences. Till at length the labour was undertaken and completed by Mr. Adams, in England, and shortly after by M. Leverrier, in France; and their efforts were crowned with success by the consequent discovery of the planet Neptune by Dr. Galle, at Berlin, and, almost, on the same day by Professor Challis, at our own Observatory.

34. When the term "mass" was first introduced, it was spoken of as the "quantity of matter" of a body. The propriety of this expression appears from the following consideration.

The mass of a body remains unaltered whatever changes of state the body undergoes, provided nothing is added to or taken from it.

If a lump of iron be heated so that its volume is increased, its mass remains unaltered. If it were removed to the Moon, where its weight would be diminished to one-sixth, or to Jupiter, where it would be increased to two and a half times its amount, its mass would remain unaltered.

If a quantity of water be converted into ice or steam, or decomposed into its component gases, the mass still remains unaltered. We see then that mass, as it has been defined, has two important properties besides that which is taken as the basis of the definition, and which are by no means consequences of the definition. The one, that which has been just described, which entitles it to the name "quantity of matter," and the other, that in the law of gravitation the attraction of two particles varies as their masses jointly.

35. We will conclude this chapter by explaining a condition which any equation expressing a relation between concrete quanti-

ties must satisfy. It must be homogeneous with respect to each of the independent concrete units that enter into it, provided no particular unit has been assumed in obtaining the equation.

Since every letter represents a number, any combination of letters will also represent a number, and any equation is algebraically correct when the number on one side of it is the same as that on the other. When, however, a letter represents a concrete magnitude expressed numerically, and the unit to which the magnitude is referred changes, the number representing the magnitude must change in the inverse ratio. Now in this subject certain of the units are defined with reference to others, so that three only are independent, and this must be considered in estimating the dimensions of any term of an equation. The independent units are those of duration, length, and mass. If the unit of length be increased n-fold the units of velocity and acceleration will also be increased n-fold. If the unit of duration be increased n-fold, the unit of velocity will be diminished to one n^{th} of its magnitude, and the unit of acceleration to one $(n^2)^{\text{th}}$. And similarly the other units will vary. This is what is meant, when velocity is said to be of one dimension in space and minus one dimension in time, and when accelerating-force is said to be of one dimension in space and minus two dimensions in time: when letters referring to these quantities occur in an equation they must be estimated in this manner; and when so estimated the dimensions of every term of the equation in each of the independent units must be the same. If this were not so, by changing one of the units for which it did not hold, we should change the several terms of the equation in different ratios, and the equality would be destroyed. But the relation expressed by the equation is true independently of any units.

Let us examine, for example, the equation

$$Mk^2\frac{d^2\theta}{dt^2} = -Wh\sin\theta.$$

M is a mass, and W which is a force is of one dimension in mass, so that the condition is satisfied with reference to the unit of mass. Again, k is a line and it enters to the second power, W is of one dimension in space, and h is also a line thus making two dimensions, and the condition is satisfied with reference to the unit of length. Again, dt enters to the second power in the denominator, and W which is a force is of minus two dimensions in time, and the condition is satisfied in this case also: $\sin\theta$ is a ratio and of no dimensions. This equation also furnishes an exception: on the left

hand side there is in the numerator $d^2\theta$, or the angle enters to the first power, on the right hand side it does not enter at all: and what is the cause of this? The equation is only true when the angle subtended by an arc equal to the radius is used as the angular unit. Any equation may be deficient in this respect, if it be restricted to one particular unit.

CHAPTER II.

36. HAVING found in the last chapter the equations of motion of a particle, the object of the present one is to explain the manner of proceeding so as to determine from them the nature of the motion which will take place.

The simplest case is when the motion is in one line and all the forces act in that line.

Let M be the mass of the particle, x its distance at the time t from some fixed point in its line of motion, v its velocity, P the resultant of all the forces which act on it: then the equations of motion are

$$M.\frac{d^2x}{dt^2} = P,$$

$$v = \frac{dx}{dt}.$$

Since, however, the mass of the particle is constant, we will divide both sides of the first equation by M, and write f instead of $\frac{P}{M}$. f is the accelerating-force of P on M. We have then

$$\frac{d^2x}{dt^2} = f,$$

$$v = \frac{dx}{dt}.$$

Now the motion of the particle will be completely determined when we are able to assign at every instant the position of the body, and the velocity with which it is moving; that is, when v and x are expressed as functions of t and known quantities. When this is done the velocity in any position may be found by eliminating t between the two equations giving v and x.

37. As a particular case suppose the body to be acted on by a constant force: in this case f is constant, and we have

$$\frac{d^2x}{dt^2} = f; \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\therefore\ v = \frac{dx}{dt} = ft + C, \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\text{and}\quad x = \tfrac{1}{2}ft^2 + Ct + C' \quad \ldots\ldots\ldots (3)$$

We have here obtained the expressions for v and x in terms of t, but these expressions contain unknown constant quantities C and C', which have been introduced by integration: to determine these it is necessary to know the circumstances of the motion at some specified time. Suppose, for instance, it is given that at a time t_1 the body is at a distance x_1 from the origin, and moving with a velocity v_1; making these substitutions in the above equation, we have

$$v_1 = f.t_1 + C$$
$$x_1 = \tfrac{1}{2}f.t_1^2 + C.t_1 + C'$$

from which to determine C and C'. When these are determined and substituted in equations (2) and (3) the values of v and x will be completely expressed in terms of t and known quantities.

The constants C and C' might be equally well determined from the position at one time and the velocity at another; or from the positions at two different times; or from other conditions under different forms. Variations of this sort constitute the difference between one problem and another.

Eliminating t between the equations (2) and (3) we have an equation between v and x. This might also be found as follows:

$$\frac{d^2x}{dt^2} = f;$$
$$\therefore \quad 2\frac{dx}{dt}\frac{d^2x}{dt^2} = 2f.\frac{dx}{dt},$$
$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = 2f.x + C'',$$
$$\text{or } v^2 = 2f.x + C''; \ldots\ldots (4)$$

and C'' must be determined from conditions analogous to those already alluded to. Such as that the velocity is v_1 when $x = x_1$, from which $v_1^2 = 2fx_1 + C''$

$$\text{and } \therefore \quad v^2 = v_1^2 + 2f.(x - x_1).$$

When a body moves in a vertical line acted on only by the attraction of the earth, within moderate distances, it may, without sensible error, be considered as acted on by a constant force. In this case, if x be measured upwards, and the accelerating force of the earth's attraction be written as usual g, we shall have $f = -g$ since the force acts in the negative direction, or tends to diminish x. The preceding equations become

$$v = -gt + C, \ldots\ldots\ldots\ldots (2)$$
$$x = -\tfrac{1}{2}gt^2 + Ct + C', \ldots\ldots\ldots (3)$$
$$\text{and } v^2 = -2gx + C'' \ldots\ldots\ldots\ldots (4).$$

Suppose now that the body is projected upwards from the origin with the velocity V, then, if t be measured from the time of projection, we have $v = V$ when $t = 0$ and $x = 0$, and therefore our equations give

$$V = C,$$
$$0 = C',$$
$$V^2 = C'';$$

and therefore $v = V - gt$(5)

$$x = Vt - \tfrac{1}{2} g t^2 \text{(6)}$$

$$v^2 = V^2 - 2gx \text{(7)}.$$

From equation (5) we see that as t increases v diminishes, that when $t = \frac{V}{g}$, v is zero, and that afterwards v becomes negative and increases in actual magnitude; also from equation (6) we see that x commencing at zero increases till $t = \frac{V}{g}$ when it equals $\frac{V^2}{2g}$, that it then diminishes, becomes zero when $t = \frac{2V}{g}$, and after this becomes negative and continues to increase in actual magnitude. From this it follows that the body will move upwards at first with a continually decreasing velocity till it attains a height $\frac{V^2}{2g}$, when it comes to rest and immediately commences moving downwards with an increasing velocity; that the time of attaining its greatest height is $\frac{V}{g}$, and the time of returning to the point from which it started is twice this. Equation (7) shews that the velocity at each point in its descending course is the same as at the corresponding point in its ascending course. It is unnecessary to dwell longer on this case; the reader will be able to frame for himself endless variations of it.

38. If the force be not constant, f will have different values at different times, and consequently will be a function of the time.

Suppose f expressed explicitly as a function of t, in this case the proceeding is as follows:

$$\frac{d^2x}{dt^2} = f,$$

$$v = \frac{dx}{dt} = \int f dt + C,$$

$$x = \iint f dt^2 + Ct + C',$$

and the determination of the constants is the same as in the preceding case.

39. If f does not depend directly on t it must depend on the position or velocity of the body, or on some circumstances external to the body, or on some combination of them. The first of these, where f depends on the position of the body, is the only case to which any general principles of solution are applicable: in the others, except in some simple cases, the equations can only be solved approximately, though it must be borne in mind that they are themselves strictly accurate.

When the force depends only on the position of the particle, it can only vary with x, and therefore f will be a function of x; let this be $f(x)$; so that the equation of motion is

$$\frac{d^2x}{dt^2} = f(x).$$

$$\therefore\ 2\ \frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = 2f(x)\cdot\frac{dx}{dt};$$

and integrating this, we have

$$v^2 = \left(\frac{dx}{dt}\right)^2 = 2\int f(x)\,dx + C.$$

This equation gives the velocity of the particle in any position.

To determine C we must know the velocity is some particular position, or have some equivalent condition.

Suppose the integration on the right hand side of the equation effected, as it generally can be; v^2 is then given as a function of x and constant quantities;

$$v^2 = \left(\frac{dx}{dt}\right)^2 = F(x);$$

and to determine the position at any time, we have

$$\frac{dt}{dx} = \pm\frac{1}{\sqrt{F(x)}},$$

where the upper or lower sign must be used according as x is increasing or diminishing with t.

Integrating this we get

$$t = \int\frac{dx}{\sqrt{F(x)}} + C';$$

or, supposing the integration on the right hand side effected, and the constant included

$$t = \phi(x),$$

and consequently $x = \psi(t)$,

and the problem is completely solved.

The different forms that may be given to $f(x)$ and the different conditions that may be assigned for the determination of the constants, are the points in which one problem differs from another. Readiness in performing the operations, in expressing analytically the conditions given for the determination of the constants, and in interpreting the results obtained is only to be acquired by the solution of a great number of problems.

The equation $\frac{d^2x}{dt^2} = f(x)$

can sometimes be solved at once as a differential equation; in this case it is generally the best method of solving the problem. We thus get at once

$$x = \psi(t),$$

and hence $v = \frac{dx}{dt} = \psi'(t),$

by differentiation: in this case $\psi(t)$ will contain two unknown constants, which have to be determined as before explained.

40. We will take the case where

$$f(x) = -\mu x,$$

μ being constant, as affording an example of both methods of solution, and being itself most important. It is the case in which the force varies directly as the distance of the body from some fixed point from which x is measured, and acts towards that point.

$$\frac{d^2x}{dt^2} = -\mu x; \quad \ldots\ldots(1)$$

$$\therefore\ 2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = -2\mu x\cdot\frac{dx}{dt};$$

$$\therefore\ v^2 = \left(\frac{dx}{dt}\right)^2 = -\mu x^2 + C.$$

Suppose now that when the particle is at a distance x_1 from the origin it is moving from it with a velocity v_1; then we have

$$v_1^2 = -\mu x_1^2 + C;$$

$$\text{and } \therefore\ v^2 = v_1^2 - \mu(x^2 - x_1^2) \ldots\ldots(2)$$

This equation gives the velocity in any position. From it we have

$$\frac{dx}{dt} = \pm\sqrt{v_1^2 - \mu(x^2 - x_1^2)},$$

we must take the upper sign because we suppose the body to be moving from the origin, and therefore $\frac{dx}{dt}$ is positive,

we have therefore $\frac{dt}{dx} = \frac{1}{\sqrt{v_1^2 + \mu x_1^2 - \mu x^2}}$,

from which $t = \frac{1}{\sqrt{\mu}} \sin^{-1} \frac{x\sqrt{\mu}}{\sqrt{v_1^2 + \mu x_1^2}} + c'$,

or altering the form of the equation,

$$x = \sqrt{\frac{v_1^2}{\mu} + x_1^2} \cdot \sin \sqrt{\mu}\,(t - c'); \ldots\ldots (3)$$

suppose that when $t = t_2$ the value of x is x_2, we then have

$$t_2 = \frac{1}{\sqrt{\mu}} \cdot \sin^{-1} \frac{x_2\sqrt{\mu}}{\sqrt{v_1^2 + \mu x_1^2}} + c',$$

which gives a value of c': we will not substitute this value because it is complicated, and the letter c' may stand as its representative.

Equation (3) shews that the value of x can never exceed $\sqrt{\frac{v_1^2}{\mu} + x_1^2}$: this is a known quantity; call it a; then equations (2) and (3) may be written

$$v^2 = \mu (a^2 - x^2), \ldots\ldots\ldots\ldots (4)$$

$$x = a \cdot \sin \sqrt{\mu}\,(t - c') \ldots\ldots (5).$$

Equation (5) shews that as t increases x will increase till $\sqrt{\mu}\,(t - c')$ first becomes an odd multiple of $\frac{\pi}{2}$; x will then be equal to a; and equation (4) shews that v will then be zero or the body will come to rest. As t goes on increasing x will diminish and the body will move in the opposite direction; when $\sqrt{\mu}\,(t - c')$ becomes a multiple of π, x will be zero and the body will be at the origin moving with a velocity $a\sqrt{\mu}$ in the negative direction: as x increases negative the velocity will decrease, and when $x = -a$ or $\sqrt{\mu}\,(t - c')$ is again an odd multiple of $\frac{\pi}{2}$, v will again be zero, and th body will be at rest at a distance a on the negative side of the origin. After this it will again move in the positive direction, and will continue moving backwards and forwards between the limits $x = a$ and $x = -a$. Any motion of this kind is called oscillatory.

We saw that the body came to rest whenever $\sqrt{\mu}\,(t - c')$ was an odd multiple of $\frac{\pi}{2}$; hence if t', t'', &c. be the successive values of t for which this is the case, we must have

$$\sqrt{\mu}\,(t'-c') = (2n+1)\frac{\pi}{2},$$

$$\sqrt{\mu}\,(t''-c') = (2n+3)\frac{\pi}{2},$$

$$\sqrt{\mu}\,(t'''-c') = (2n+5)\frac{\pi}{2},$$

$$\therefore\ t''-t' = t'''-t'' = \&c. = \frac{\pi}{\sqrt{\mu}}.$$

This result is curious, inasmuch as it shews that the time of moving from rest to rest is independent of the distance between the points of rest.

The time from rest to rest is called the time of an oscillation, the distance between the points of rest is called the amplitude of the oscillation. The peculiarity of this particular law of force is that the time of oscillation is independent of the amplitude.

41. We will now apply the other method of solving the equation

$$\frac{d^2x}{dt^2} + \mu x = 0 \ldots\ldots (1).$$

The integral of this is known to be

$$x = A\sin(\sqrt{\mu}\,.\,t + B), \ldots\ldots (2)$$

where A and B are unknown constants.

Differentiating (2) we have

$$v = \frac{dx}{dt} = A\sqrt{\mu}\,.\cos(\sqrt{\mu}\,.\,t + B), \ldots\ldots (3)$$

Equations (2) and (3) are exactly of the same form as those obtained before for x and v. The constants may be determined in a similar manner and the interpretation is exactly the same.

Whenever an equation appears under the preceding form, this latter method should always be adopted in integrating it. The equation is one which presents itself continually, and should be familiar to every one.

If the equation is of the form

$$\frac{d^2x}{dt^2} - \mu x = 0,$$

which corresponds to a repulsive force varying according to the same law, the integral is

$$x = C_1\epsilon^{\sqrt{\mu}.t} + C_2\epsilon^{-\sqrt{\mu}.t},$$

and the motion will not be oscillatory at all.

42. Let us now consider the case where the force varies as the n^{th} power of the distance; in this case

$$\frac{d^2x}{dt^2} = -\mu x^n; \ldots\ldots(1)$$

$$\therefore 2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = -2\mu x^n\frac{dx}{dt};$$

$$\therefore v^2 = \left(\frac{dx}{dt}\right)^2 = C - \frac{2\mu x^{n+1}}{n+1}.$$

If $v = v_1$ when $x = a$, we have

$$v_1^2 = C - \frac{2\mu a^{n+1}}{n+1};$$

$$\therefore v^2 - v_1^2 = \frac{2\mu}{n+1}(a^{n+1} - x^{n+1}) \ldots\ldots(2)$$

which gives the velocity in any position.

The next integration which would give the relation between x and t cannot generally be effected.

If v_1 and a are both positive the body is then moving from the origin, and equation (2) shews that as x increases v diminishes; let c be the value of x that makes $v = 0$, then c will be given by the equation

$$v_1^2 + \frac{2\mu}{n+1}(a^{n+1} - c^{n+1}) = 0,$$

and equation (2) may be replaced by

$$v^2 = \frac{2\mu}{n+1}(c^{n+1} - x^{n+1}) \ldots\ldots\ldots\ldots(3).$$

There is a peculiarity about this expression for the velocity when n is even which is worthy of attention, as it illustrates a point requiring great care when physical conditions are to be expressed analytically.

When $x = c$, $v = 0$, and the particle is at rest at a distance c from the origin, acted on by a force tending towards the origin; it will therefore commence moving in that direction, and its velocity at any distance will be given by equation (3).

First let $n + 1$ be positive: then if V be the velocity of the body when it reaches the origin,

$$V^2 = \frac{2\mu c^{n+1}}{n+1}.$$

So far the force has been increasing the velocity of the particle, its direction now changes, since it is supposed to act towards the point

chosen as origin, and it will therefore tend to diminish the velocity; and since the intensity of the force is the same at equal distances on opposite sides of the origin, it is to be expected that the velocity will be destroyed by exactly the same steps by which it was generated. This conclusion follows from the similarity of the action of forces when they accelerate and retard the motion of a particle. It also follows from the equation

$$v^2 = \frac{2\mu}{n+1}(c^{n+1} - x^{n+1}) \ldots\ldots\ldots\ldots (3)$$

when $n+1$ is even, since in this case the equation gives the same value to v for values of x equal in magnitude and of opposite signs, and therefore as x increases negative the velocity will gradually diminish, being always the same as it was at an equal distance on the opposite side, and will ultimately equal zero when $x = -c$.

When, however, $n+1$ is odd, negative values of x substituted in equation (3) give for v values greater than V. Now it is impossible that a force acting in a direction opposite to the motion of a particle can increase its velocity: therefore equation (3) cannot correctly represent the velocity on the negative side of the origin. Referring to the equation

$$\frac{d^2x}{dt^2} = -\mu x^n \ldots\ldots\ldots\ldots (1)$$

from which we started, we find that $-\mu x^n$, the analytical expression for the force, represents correctly a force tending towards the origin, both for positive and negative values of x, when n is odd; but that when n is even it represents a force tending towards the origin for positive values of x only, and therefore conclusions derived from it will not be true for negative values of x. For negative values of x the correct equation is

$$\frac{d^2x}{dt^2} = \mu x^n,$$

$$\text{from which} \quad v^2 = \frac{2\mu x^{n+1}}{n+1} + C,$$

$$\text{and making} \quad v = -\sqrt{\frac{2\mu c^{n+1}}{n+1}},$$

when $x = 0$, which amounts to supposing the particle to pass through the origin in the negative direction with the velocity V, we have

$$v^2 = \frac{2\mu}{n+1}(c^{n+1} + x^{n+1}) \ldots\ldots\ldots\ldots (4)$$

as the equation to be used in this case instead of equation (3).

This equation shews that the velocity gradually diminishes as x changes from 0 to $-c$ when it equals zero, consistently with what takes place when n is odd.

Now let n be negative, and $= -m$. Equation (3) becomes

$$v^2 = \frac{2\mu}{m-1}\left(\frac{1}{x^{m-1}} - \frac{1}{c^{m-1}}\right) \ldots\ldots\ldots\ldots (5)$$

and we see that the velocity increases without limit as the body approaches the origin. This was to be expected since the force which produces the motion itself increases without limit. As in the previous case, when $m-1$ is even, this equation shews that the velocity decreases, as the body recedes from the origin in the negative direction, till it comes to rest at a distance c. When $m-1$ is odd, negative values of x substituted in (5) give impossible values to v. The explanation in this case is the same as before: when m is odd $-\frac{\mu}{x^m}$ represents the force correctly on both sides of the origin, when m is even $-\frac{\mu}{x^m}$ must be used for positive values of x, and $\frac{\mu}{x^m}$ for negative values, and the velocity on the negative side of the origin will be given by the equation

$$v^2 = -\frac{2\mu}{m-1}\left(\frac{1}{x^{m-1}} + \frac{1}{c^{m-1}}\right).$$

In all these cases then the body will pass through the origin and move in the negative direction, losing its velocity at the same rate at which it acquired it, and will come to rest when $x = -c$. It will then move towards the origin and pass through it as before, continuing to oscillate between the limits $x = c$ and $x = -c$.

43. In the cases of rectilinear motion which have hitherto been considered, the forces which acted on the particle were either constant, or some functions of the position of the particle.

The force may also be some function of the velocity with which the particle is moving; that is, the equation of motion which has hitherto always been of the form

$$\frac{d^2x}{dt^2} = f(x)$$

may now become $\frac{d^2x}{dt^2} = f\left(\frac{dx}{dt}\right)$,

or $\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right)$.

These equations when they occur must be solved by the ordinary methods applied to differential equations.

We will solve two or three of them to exhibit the methods.

Let the equation of motion be

$$\frac{d^2x}{dt^2} = -\mu\left(\frac{dx}{dt}\right)^n, \quad \ldots\ldots\ldots\ldots(1)$$

that is, suppose the only force which acts on the body to be a retarding force varying as the n^{th} power of the velocity. The above equation may be written

$$\frac{1}{v^n}\cdot\frac{dv}{dt} = -\mu.$$

The integral of which is

$$\frac{1}{v^{n-1}} = (n-1)\mu t + C.$$

Let $v = v_1$ when $t = 0$, then

$$\frac{1}{v_1^{n-1}} = C,$$

$$\text{and } \frac{1}{v^{n-1}} - \frac{1}{v_1^{n-1}} = (n-1)\mu t.$$

This equation shews that as t increases the velocity will continually diminish, but will never be actually zero. From this equation

$$v^{n-1} = \frac{v_1^{n-1}}{1 + v_1^{n-1}.(n-1)\mu t};$$

$$\text{and therefore } \frac{dx}{dt} = v = \frac{v_1}{\{1 + v_1^{n-1}.(n-1)\mu t\}^{\frac{1}{n-1}}},$$

$$\text{from which } C + x = \frac{\{1 + v_1^{n-1}.(n-1)\mu t\}^{\frac{n-2}{n-1}}}{\mu(n-2)v_1^{n-2}},$$

which gives x as a function of t.

The above solution becomes nugatory when $n = 1$; in this case the solution of the differential equation takes a different form; we have

$$\frac{d^2x}{dt^2} = -\mu\frac{dx}{dt}; \quad \ldots\ldots\ldots\ldots (1)$$

$$\therefore \frac{dx}{dt} = C - \mu x,$$

let v_1 be the velocity when $x = a$, then

$$v_1 = C - \mu a,$$

$$\therefore \frac{dx}{dt} - v_1 = \mu(a-x); \ldots\ldots\ldots(2)$$

$$\text{or } \frac{1}{\mu a + v_1 - \mu x} \cdot \frac{dx}{dt} = 1;$$

$$\therefore \log\{\mu(a-x)+v_1\} = -\mu t + C,$$

let $t=0$ when $x=a$, then

$$C = \log v_1,$$

$$\text{and } \mu(a-x)+v_1 = v_1 \epsilon^{-\mu t}, \ldots\ldots\ldots(3)$$

which gives x as a function of t.

Equation (1) might have been integrated as follows:

$$\frac{1}{v} \cdot \frac{dv}{dt} = -\mu;$$

$$\therefore \log v = -\mu t + C,$$

$$\log v_1 = C;$$

$$\therefore v = v_1 \epsilon^{-\mu t} \ldots\ldots\ldots\ldots\ldots(4).$$

44. Cases of motion of this sort are more curious than useful. The only case known to exist in nature of a force depending on the velocity of the moving body, is when the body moves in a fluid, or, as it is commonly called, a resisting medium. In this case it is found that the fluid exerts a pressure on the particle in a direction opposite to that in which it is moving, and varying directly as the square of the velocity. If the body is acted on by no other force the solution of this is included in the general case just discussed. Let us consider the case where the body is acted on by a constant force as well.

The equation of motion is

$$\frac{d^2x}{dt^2} = f - kv^2, \ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{or } \frac{dv}{dt} = f - kv^2;$$

$$\therefore \frac{1}{f-kv^2} \cdot \frac{dv}{dt} = 1;$$

$$\therefore \frac{1}{2\sqrt{fk}} \cdot \log \frac{\sqrt{f}+v\sqrt{k}}{\sqrt{f}-v\sqrt{k}} = t + C.$$

Let $v = v_1$ when $t=0$, then

$$\frac{1}{2\sqrt{fk}} \cdot \log \frac{\sqrt{f}+v\sqrt{k}}{\sqrt{f}-v\sqrt{k}} \cdot \frac{\sqrt{f}-v_1\sqrt{k}}{\sqrt{f}+v_1\sqrt{k}} = t;$$

$$\text{or } \frac{\sqrt{f}+v\sqrt{k}}{\sqrt{f}-v\sqrt{k}}=\frac{\sqrt{f}+v_1\sqrt{k}}{\sqrt{f}-v_1\sqrt{k}}\cdot\epsilon^{2\sqrt{fk}\cdot t};$$

$$\therefore\ v=\sqrt{\frac{f}{k}}\cdot\frac{\mu\epsilon^{2\sqrt{fk}\cdot t}-1}{\mu\epsilon^{2\sqrt{fk}\cdot t}+1}\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\text{where } \mu=\frac{\sqrt{f}+v_1\sqrt{k}}{\sqrt{f}-v_1\sqrt{k}};$$

we may also solve equation (1) as follows:

$$\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}=v\cdot\frac{dv}{dx}.$$

So that (1) becomes

$$v\frac{dv}{dx}+kv^2=f.$$

Let $2w=v^2$, then $\frac{dw}{dx}=v\frac{dv}{dx}$;

$$\text{and } \therefore\ \frac{dw}{dx}+2kw=f;$$

$$\therefore\ w=\frac{f}{2k}+C\cdot\epsilon^{-2kx},$$

$$\text{or } v^2=\frac{f}{k}+2C\cdot\epsilon^{-2kx}.$$

Let $v=v_1$ when $x=a$, then

$$f-kv^2=(f-kv_1^2)\,\epsilon^{-2k(x-a)}\ldots\ldots(3).$$

Equations (2) and (3) comprise the complete solution of the problem; they give two relations between v, t and x.

It appears from an examination of either of these equations that the velocity continually approaches $\sqrt{\frac{f}{k}}$ as its limit; if the original velocity v_1 is greater than this it will continue diminishing, but will never become so small as $\sqrt{\frac{f}{k}}$; if the original velocity is less than $\sqrt{\frac{f}{k}}$ it will continually increase, but never become so great as $\sqrt{\frac{f}{k}}$; the name "terminal velocity" has been given to $\sqrt{\frac{f}{k}}$.

In motion of this kind considered as motion in a resisting medium, the resistance is always in the direction opposite to that in which the body is moving, so that if the forces which act on the body are such as cause it to come to rest and then commence moving in the opposite direction, the direction of the resistance will change. In any case of this sort the problem must be separated into two, the one commencing where the other left off, at the point where the discontinuity takes place. The following is an instance.

45. Let a constant force act on the body in a direction opposite to that in which the body is moving. The equation of motion is

$$\frac{d^2x}{dt^2} = -f - kv^2,$$

$$\text{or, } \frac{dv}{dt} = -f - kv^2.$$

The integral of this is

$$\frac{1}{\sqrt{fk}} \cdot \tan^{-1}\sqrt{\frac{k}{f}} \cdot v = C - t.$$

Let v_1 be the initial velocity, then

$$\tan^{-1}\sqrt{\frac{k}{f}} \cdot v - \tan^{-1}\sqrt{\frac{k}{f}} \cdot v_1 = -\sqrt{fk} \cdot t \ldots\ldots\ldots(1).$$

As t increases v diminishes, and becomes zero when

$$t = \frac{1}{\sqrt{fk}} \tan^{-1}\sqrt{\frac{k}{f}} \cdot v_1.$$

Adopting the other method of integration, we have

$$v\frac{dv}{dx} + kv^2 = -f;$$

$$\therefore \frac{dw}{dx} + 2kw = -f;$$

$$\therefore w = -\frac{f}{2k} + C \cdot \epsilon^{-2kx};$$

$$\therefore f + kv^2 = 2kC \cdot \epsilon^{-2kx}.$$

Let x be measured from the starting point, then

$$f + kv_1^2 = 2kC,$$

$$\text{and } f + kv^2 = (f + kv_1^2)\,\epsilon^{-2kx} \ldots\ldots\ldots (2),$$

which gives the velocity of the body in any position.

By eliminating v between (1) and (2) we may obtain a relation between x and t. If in equation (2) we put $v = 0$, we have

$$\epsilon^{2kx} = \frac{kv_1^2}{f} + 1 \ldots\ldots\ldots(3),$$

which gives the distance to which the body will move before it comes to rest. After this it will commence moving in the opposite direction, and the equations which determine the velocity and position in terms of the time, will be deduced from the case previously discussed, by putting $v_1 = 0$; we thus get

$$v = \sqrt{\frac{f}{k}} \cdot \frac{\epsilon^{2\sqrt{fk}.t} - 1}{\epsilon^{2\sqrt{fk}.t} + 1} \quad \ldots\ldots\ldots(4)$$

$$\text{and } v^2 = \frac{f}{k}(1 - \epsilon^{-2kx}), \quad \ldots\ldots\ldots(5)$$

in equation (5) x is measured in the new direction of motion from the point where the body came to rest.

If a be the value of x which satisfies equation (3), equation (5) may be written

$$v^2 = \frac{f}{k}\{1 - \epsilon^{-2k(a-x)}\}, \quad \ldots\ldots\ldots(6)$$

in which x is measured from the original starting point in the direction of projection.

When $x = 0$ in this equation we have

$$v^2 = \frac{f}{k}(1 - \epsilon^{-2ka}),$$

or, substituting for a its value given by equation (3)

$$v^2 = v_1^2 \cdot \frac{f}{f + kv_1^2},$$

which gives the velocity with which the body returns to the point from which it started.

This is the case of a body projected upwards in the air: writing g for f the results may be stated as follows.

A body projected upwards with the velocity v_1 will rise to a height a given by the equation

$$g \cdot \epsilon^{2ka} = g + kv_1^2,$$

in the time t_1 given by the equation

$$t_1\sqrt{gk} = \tan^{-1}\sqrt{\frac{k}{g}} \cdot v_1,$$

it will then return in the time t_2 given by the equation

$$\epsilon^{2\sqrt{gk}.t_2} = \frac{\sqrt{g + kv_1^2} + v_1\sqrt{k}}{\sqrt{g + kv_1^2} - v_1\sqrt{k}},$$

and reach the starting point with the velocity $v_1\sqrt{\frac{g}{g + kv_1^2}}$.

These examples will be sufficient to shew how cases of motion of this kind are to be treated; it will be seen at once that the difficulties they present are entirely analytical.

Before leaving the subject of resistances, however, we will make one remark which will be understood at once by any one familiar with the nature of fluid pressure.

The actual force of resistance exercised by the fluid will be the same for all bodies of the same size and shape. Consequently the accelerating, or rather retarding, force of the resistance will vary inversely as the mass of the body in motion. For bodies of the same shape and size may have very different masses if composed of different materials. Now it is with the retarding force that we are concerned in dynamics. We draw attention to it in this case because it is apt to be overlooked, from the fact that most of the forces with which we are concerned in dynamics vary directly as the masses of the bodies on which they act, and consequently their accelerating forces are independent of those masses.

46. The preceding examples will be sufficient to shew how any problem on the rectilinear motion of a particle is to be treated. The first step is to express analytically the forces which act on the particle, and to form the equation of motion; the next step is to integrate this equation; the third is to express analytically the conditions given for determining the constants introduced by integration; and the last and by no means least important is to interpret the equations thus obtained, that is, to examine them for different values of the variable quantities which enter into them, and so to trace the motion of the particle through any peculiarities it may present.

47. Let us now proceed to the case of curvilinear motion, and first, for the sake of simplicity, we will consider motion in one plane.

The equations we have obtained are

$$M \cdot \frac{d^2x}{dt^2} = X,$$

$$M \cdot \frac{d^2y}{dt^2} = Y.$$

If $X=0$ and $Y=0$, the particle will move on in a straight line by the first law of motion.

Let $X=0$ and $Y=-Mg$. This will be the case of a body projected from the surface of the earth and acted on by gravity, the

axis of x being measured horizontally and the axis of y vertically upwards. The equations of motion are

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= 0,\\ \frac{d^2y}{dt^2} &= -g.\end{aligned}\right\}\ldots\ldots\ldots(1)$$

Integrating these we have

$$\frac{dx}{dt} = C,$$

$$\frac{dy}{dt} = C' - gt.$$

Suppose the body to be projected from the origin with a velocity V, in a direction making an angle α with the axis of x, and let t be measured from that instant; then we have

$$\left.\begin{aligned}\frac{dx}{dt} &= V\cos\alpha,\\ \frac{dy}{dt} &= V\sin\alpha - gt.\end{aligned}\right\}\ldots\ldots\ldots(2)$$

And integrating again, we have

$$x = V\cos\alpha\,.\,t + C'',$$

$$y = V\sin\alpha\,.\,t - \tfrac{1}{2}gt^2 + C''',$$

or since x and y are zero when $t = 0$, we have

$$\left.\begin{aligned}x &= V\cos\alpha\,.\,t\\ y &= V\sin\alpha\,.\,t - \tfrac{1}{2}gt^2\end{aligned}\right\}\ldots\ldots\ldots(3).$$

Equations (2) give the resolved parts of the velocity, and equations (3) the position at any time.

From (2) $$\frac{dy}{dx} = \tan\alpha - \frac{gt}{V\cos\alpha},$$

which gives the direction of motion,

and $$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{V^2 - 2gtV\sin\alpha + g^2t^2},$$

which gives the velocity at any time.

Eliminating t between equations (3) we have

$$y = \tan\alpha\,.\,x - \frac{gx^2}{2V^2\cos^2\alpha},$$

a relation between the co-ordinates of the particle independent of the time, and therefore the equation to its path.

This equation shews that the curve in which the particle moves is a parabola whose axis is vertical, and concavity downwards; putting it in the form

$$\left(x-\frac{V^2\sin\alpha\cos\alpha}{g}\right)^2=\frac{2V^2\cos^2\alpha}{g}\left(\frac{V^2\sin^2\alpha}{2g}-y\right),$$

we see that the co-ordinates of the vertex are

$$\frac{V^2\sin\alpha\cos\alpha}{g} \text{ and } \frac{V^2\sin^2\alpha}{2g},$$

and the latus rectum is $\dfrac{2V^2\cos^2\alpha}{g}$.

The height of the directrix above the origin is

$$\frac{V^2\sin^2\alpha}{2g}+\frac{V^2\cos^2\alpha}{2g} \text{ which } =\frac{V^2}{2g}.$$

If the body fell from rest vertically through this height it would acquire the velocity V; and since any point of the body's course may be considered the starting point, it follows that the velocity at any point in the path is that which the body would acquire in falling vertically from the directrix to that point.

We see from equations (2) that the resolved part of the velocity in a horizontal direction is constant throughout the motion, and that the vertical component varies as it would do if the body were moving in a vertical line. This consideration is useful in discussing many points in the motion of projectiles.

From these equations there will be no difficulty in obtaining the "range" of the projectile, that is, the distance of the point at which it will strike a horizontal or inclined plane passing through the point of projection; and the time that will be occupied in it, which is called "the time of flight."

Sometimes it will happen that V and α are not given, and have to be determined by the conditions of the problem, as for instance, to find V and α, that the path may pass through two given points.

By varying the conditions of this sort, an endless number of problems may be formed, all of which may be solved by the application of the above equations, either in their present or in a modified form.

If instead of $X=0$ we had X constant, the case would not be altered, for the two constant forces X and Y would be equivalent to some constant force in some constant direction; and no force would act in a direction at right angles to this.

48. It is unnecessary to consider the case of X and Y being functions of t explicitly as this never occurs in nature.

Let then X and Y depend on the position of the particle, and first let

X be a function of (x) alone

and Y a function of (y) alone.

In this case we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= f(x)\\ \frac{d^2y}{dt^2} &= F(y)\end{aligned}\right\}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\therefore\ 2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = 2f(x)\ \frac{dx}{dt},$$

$$2\frac{dy}{dt}\cdot\frac{d^2y}{dt^2} = 2F(y)\frac{dy}{dt};$$

$$\left.\begin{aligned}\therefore \left(\frac{dx}{dt}\right)^2 &= 2\int f(x)dx + C\\ \left(\frac{dy}{dt}\right)^2 &= 2\int F(y)dy + C'\end{aligned}\right\} \ldots\ldots\ldots (2).$$

These two equations give the resolved parts of the velocity, and therefore the actual velocity and direction in any position. C and C' must be determined from the velocity and direction in some known position.

From equations (2) we have

$$\left(\frac{dy}{dx}\right)^2 = \frac{2\int F(y)dy + C'}{2\int f(x)dx + C}, \quad \ldots\ldots (3)$$

which gives the direction of motion at any instant, or the position of the tangent to the path.

If we can integrate equations (2), we shall obtain two relations, between x and t, and between y and t; each of these will contain a new constant which must be determined from the position of the particle at some instant: if between these two equations we eliminate t, we shall obtain a relation between x and y which will be the equation to the path of the particle.

The equation to the path may also be found at once by integrating equation (3). The constant in this case will be determined from any corresponding values of x and y, or, in other words, any point through which the curve passes. We may remark that equation (3) will always be integrable when equations (2) are, and never when they are not.

Supposing all the integrations effected, we have $\frac{dx}{dt}$ and $\frac{dy}{dt}$ expressed as functions of x and y; we also have x and y as functions of t; we may therefore by elimination find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as functions of t; we shall then know the position, velocity, and direction of motion of the particle at every instant, and the problem will be completely solved.

49. As an example,

$$\text{let } X = -\mu x, \quad Y = -\mu y,$$

so that our equations are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -\mu x \\ \frac{d^2y}{dt^2} &= -\mu y \end{aligned}\right\}; \ldots\ldots (1)$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = C_1 - \mu x^2,$$

$$\left(\frac{dy}{dt}\right)^2 = C_2 - \mu y^2,$$

when $x = h$, and $y = k$, let V be the velocity of the particle and α the angle which the direction of motion makes with the axis of x,

then we have $V^2 \cos^2\alpha = C_1 - \mu h^2$,

$$V^2 \sin^2\alpha = C_2 - \mu k^2,$$

from which $\left(\frac{dx}{dt}\right)^2 = V^2 \cos^2\alpha + \mu h^2 - \mu x^2$,

$$\left(\frac{dy}{dt}\right)^2 = V^2 \sin^2\alpha + \mu k^2 - \mu y^2;$$

$$\therefore \frac{dt}{dx} = \frac{\pm 1}{\sqrt{V^2 \cos^2\alpha + \mu h^2 - \mu x^2}},$$

$$\frac{dt}{dy} = \frac{\pm 1}{\sqrt{V^2 \sin^2\alpha + \mu k^2 - \mu y^2}}.$$

Taking the upper sign in each case and integrating, we have

$$\sqrt{\mu} \,.\, t + C_3 = \sin^{-1} \frac{x\sqrt{\mu}}{\sqrt{V^2 \cos^2\alpha + \mu h}},$$

$$\sqrt{\mu} \,.\, t + C_4 = \sin^{-1} \frac{y\sqrt{\mu}}{\sqrt{V^2 \sin^2\alpha + \mu k^2}};$$

and therefore $\sqrt{\mu} \,.\, x = \sqrt{V^2 \cos^2\alpha + \mu h^2} \sin(\sqrt{\mu} \,.\, t + C_3)$,

$$\sqrt{\mu} \,.\, y = \sqrt{V^2 \sin^2\alpha + \mu k^2} \,.\, \sin(\sqrt{\mu} \,.\, t + C_4).$$

If in these equations we substitute corresponding values of x, y, and t we shall determine C_3 and C_4, and x and y will then be completely known as functions of t, and also $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Also by eliminating t we should obtain an equation between x and y, which would be the equation to the curve described by the body. It will be found on elimination that this curve is an ellipse whose centre is the origin.

We might have integrated equations (1) at once, as linear equations, and that would in this case have been the easier method of solution though not so generally applicable. We will illustrate this method with particular values of the constants h, k and a.

Suppose the particle to be projected from a point in the axis of x at a distance a from the origin in a direction parallel to the axis of y, with a velocity V. The equations are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \mu x = 0 \\ \frac{d^2y}{dt^2} + \mu y = 0 \end{aligned}\right\} \ldots\ldots (1).$$

The integrals of which are

$$x = A_1 \cos(\sqrt{\mu} \,.\, t + B_1),$$
$$y = A_2 \cos(\sqrt{\mu} \,.\, t + B_2),$$

and therefore $\frac{dx}{dt} = -A_1 \sqrt{\mu} \sin(\sqrt{\mu} \,.\, t + B_1)$,

$$\frac{dy}{dt} = -A_2 \sqrt{\mu} \sin(\sqrt{\mu} \,.\, t + B_2):$$

when $t = 0$, we have

$$x = a,\ y = 0,\ \frac{dx}{dt} = 0,\ \frac{dy}{dt} = V;$$

making these substitutions, we have the equations

$$a = A_1 \cos B_1, \ \ldots\ldots\ldots (2)$$
$$0 = A_2 \cos B_2, \ \ldots\ldots\ldots (3)$$
$$0 = -A_1 \sqrt{\mu} \sin B_1, \ldots\ldots (4)$$
$$V = -A_2 \sqrt{\mu} \sin B_2 \ldots\ldots (5)$$

From (2) and (4), we have

$$B_1 = 0,\ A_1 = a.$$

From (3) and (5), we have

$$B_2 = \frac{\pi}{2},\ A_2 = -\frac{V}{\sqrt{\mu}}.$$

Substituting these values we have

$$\left.\begin{aligned} x &= a\cos(\sqrt{\mu}\,.\,t) \\ y &= \frac{V}{\sqrt{\mu}}\sin(\sqrt{\mu}\,.\,t) \end{aligned}\right\}, \quad \ldots\ldots(6)$$

$$\left.\begin{aligned} \frac{dx}{dt} &= -a\sqrt{\mu}\sin(\sqrt{\mu}\,.\,t) \\ \frac{dy}{dt} &= V\cos(\sqrt{\mu}\,.\,t) \end{aligned}\right\}; \quad \ldots\ldots(7)$$

and eliminating t between equations (6) we have

$$\frac{x^2}{a^2} + \frac{\mu y^2}{V^2} = 1 \ldots\ldots\ldots(8).$$

The equation to an ellipse whose semiaxes are a and $\left(\frac{V}{\sqrt{\mu}}\right)$; or putting ($b$) for the last, we have

$$x = a\cos\sqrt{\mu}t,$$

$$y = b\sin\sqrt{\mu}t,$$

$$\frac{dx}{dt} = -a\sqrt{\mu}\sin\sqrt{\mu}t,$$

$$\frac{dy}{dt} = b\sqrt{\mu}\cos\sqrt{\mu}t,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In these equations t is the time from the extremity of that axis which coincides with the axis of x.

These equations are very often useful and are worth remembering.

In the preceding example the resultant of the forces X and Y always passes through the origin, and is equal to μ times the distance from the origin. This is a particular case of a class of forces called central forces which will shortly occupy a large share of our attention.

The example we have given will be sufficient to illustrate the method of proceeding in problems of this class. In particular cases, artifices will frequently suggest themselves which will render the solution much easier; these however are best acquired by practice.

50. We will now enter upon the consideration of the case where X and Y are each functions of both x and y.

The above method of solution evidently will not apply, nor can we offer any that will be generally applicable. The following however will often be found useful.

The equations are

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= X\\ \frac{d^2y}{dt^2} &= Y\end{aligned}\right\}; \quad \ldots\ldots\ldots(1)$$

$$\therefore\ 2\left(\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}+\frac{dy}{dt}\cdot\frac{d^2y}{dt^2}\right)=2\left(X\frac{dx}{dt}+Y\frac{dy}{dt}\right),$$

$$\text{or } d.v^2=2(X\,dx+Y\,dy)\ldots\ldots\ldots(2)$$

When the right hand side of this equation is a perfect differential of x and y considered as independent variables, we have by integration

$$v^2=c+f(x,y).$$

If V be the velocity when $x=a$, and $y=b$,

$$V^2=c+f(a,b);$$

$$\therefore\ v^2-V^2=f(x,y)-f(a,b)$$

which shews that the change in the velocity in passing from any one point to any other depends only on the co-ordinates of those points, and is independent of the path by which the body has passed from one to the other; that is, if from any point (a, b) we can project the body with a velocity V in any number of different ways so as to pass through the point (x, y), it will in each case pass through it with the velocity v.

Again, referring to equations (1), we have

$$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=xY-yX.$$

If the right-hand side of this equation is constant and equal to c we have by integration

$$x\frac{dy}{dt}-y\frac{dx}{dt}=c.t+c'.$$

The only case in which this method of integration is useful is when $c=0$, in this case the resultant of the forces X, Y passes through the origin, and we have

$$x\frac{dy}{dt}-y\frac{dx}{dt}=h.$$

If A be the area swept out by the line joining the origin and the particle, we have

$$2\frac{dA}{dt}=x\frac{dy}{dt}-y\frac{dx}{dt}=h;$$

$$\therefore\ 2A=ht+C,$$

or if A be reckoned from the time when $t = 0$,

$$2A = ht,$$

that is, the area described in any time is proportional to the time of describing it.

This is true whenever the resultant force which acts on the particle passes through the origin; that is, through any fixed point which may be taken as the origin.

These equations when obtained are only the first integrals of the equations of motion; the second must be obtained by artifices suggested by the forms of the equations in each particular case.

51. There is a class of problems the inverse of those we have been considering, viz. where it is required to find what must be the forces under whose action a particle would move in a certain specified manner.

If the co-ordinates x, y of the particle were given as functions of t we should obtain the resolved parts of the force at once by two differentiations, this would be the simplest form of the problem. A more common form of it is to find the force under the action of which a particle would move in a certain given curve.

$$\text{Let } F(x, y) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be the equation to the given curve: differentiating this equation we have

$$F'(x)\frac{dx}{dt} + F'(y)\frac{dy}{dt} = 0. \ldots\ldots\ldots (2)$$

If we differentiate this again, we have

$$F'(x)\frac{d^2x}{dt^2} + F'(y)\frac{d^2y}{dt^2} + F''(x)\left(\frac{dx}{dt}\right)^2 + F''(y)\left(\frac{dy}{dt}\right)^2 + 2F''(x, y)\frac{dx}{dt}\cdot\frac{dy}{dt} = 0 \ldots\ldots\ldots\ldots (3)$$

Now our object is to find $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$, and we see that from equation (1) we are able to obtain only a single relation (3) between these quantities and others: some other condition therefore is necessary to render the problem definite. We might, for instance, assign one of the forces arbitrarily as a function of x and y, and equation (3) would then enable us to determine the other: thus let

$$\frac{d^2x}{dt^2} = \phi(x, y),$$

eliminating y between this and (1) we obtain

$$\frac{d^2x}{dt^2} = \psi(x),$$

from which we may find $\frac{dx}{dt}$.

Substituting this value in equation (2) we obtain $\frac{dy}{dt}$, and $\frac{d^2y}{dt^2}$ may then be found by differentiation, or from equation (3).

In this case we had to integrate $\frac{d^2x}{dt^2}$ to determine $\frac{dx}{dt}$; and generally, whatever condition is given in addition to the equation to the curve, it will be necessary to express it in the form

$$f\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}\right) = 0; \ldots\ldots\ldots (4)$$

that is, in the form of an equation involving one or both of the first differential coefficients and neither of the second differential coefficients.

From (1), (2) and (4) we can always deduce equations of the form

$$\left(\frac{dx}{dt}\right)^2 = \phi(x),$$

$$\left(\frac{dy}{dt}\right)^2 = \psi(y),$$

from which we obtain

$$\frac{d^2x}{dt^2} = \frac{1}{2}\phi'(x),$$

$$\frac{d^2y}{dt^2} = \frac{1}{2}\psi'(y).$$

The resolved parts of the force are then known.

Problems of this kind seldom present any difficulty, as the operations consist for the most part of differentiation, and not of integration.

52. For example, let it be required to find under the action of what force a body would describe an ellipse with uniform velocity.

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (1)

be the equation to the ellipse, v the constant velocity, then

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = v^2 \ldots\ldots\ldots (2)$$

from equation (1) $\frac{x}{a^2} \cdot \frac{dx}{dt} + \frac{y}{b^2} \cdot \frac{dy}{dt} = 0$ (3)

Eliminating y and $\frac{dy}{dt}$ from these three equations we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{a^2v^2(a^2-x^2)}{a^4-(a^2-b^2)x^2} \quad \ldots\ldots\ldots (4)$$

Differentiating (4) we have

$$\frac{X}{M} = \frac{d^2x}{dt^2} = -\frac{a^4b^2v^2x}{\{a^4-(a^2-b^2)x^2\}^2}.$$

Similarly we may obtain

$$\frac{Y}{M} = \frac{d^2y}{dt^2} = -\frac{a^2b^4v^2y}{\{b^4-(b^2-a^2)y^2\}^2}.$$

X and Y are the resolved parts of the force parallel to the axes.

Since $$\frac{Y}{X} = \frac{b^2y}{a^2x} \cdot \left\{\frac{a^4-(a^2-b^2)x^2}{b^4-(b^2-a^2)y^2}\right\}^2 = \frac{a^2y}{b^2x},$$

we see that the resultant of X and Y coincides with the normal to the ellipse at every point.

Let $a = b$ and the ellipse be a circle; then

$$X = -M \cdot \frac{v^2}{a^2}x,$$

$$Y = -M \cdot \frac{v^2}{a^2}y,$$

and the resultant, which acts in the normal, and therefore always passes through the centre

$$= \sqrt{X^2+Y^2} = M\frac{v^2}{a};$$

that is, a body will describe a circle under the action of a constant force $M\frac{v^2}{a}$ tending to the centre.

We may shew, generally, that when a body moves with uniform velocity, the resultant force on it acts in direction of the normal: for since the velocity is uniform

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \text{a constant quantity};$$

$$\therefore \frac{dx}{dt}\cdot\frac{d^2x}{dt^2} + \frac{dy}{dt}\cdot\frac{d^2y}{dt^2} = 0;$$

$$\therefore \frac{Y}{X} = \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}} = -\frac{dx}{dy},$$

which proves the proposition.

53. It is sometimes convenient to employ polar co-ordinates in the solution of problems.

In this case the force is resolved in direction of, and perpendicular to the radius vector.

We will conclude this Chapter by making the requisite transformations.

Let P be the part of the force tending to the pole, T the part perpendicular to that direction,

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

then we have

$$P = -X\cos\theta - Y\sin\theta,$$

$$T = -X\sin\theta + Y\cos\theta.$$

The equations of motion are $\frac{d^2x}{dt^2} = X$,

$$\frac{d^2y}{dt^2} = Y;$$

$$\therefore \cos\theta\frac{d^2x}{dt^2} + \sin\theta\frac{d^2y}{dt^2} = -P,$$

$$\cos\theta\frac{d^2y}{dt^2} - \sin\theta\frac{d^2x}{dt^2} = T.$$

$$\text{Now, } \frac{dx}{dt} = \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt},$$

$$\frac{dy}{dt} = \sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt};$$

$$\therefore \frac{d^2x}{dt^2} = \cos\theta\frac{d^2r}{dt^2} - 2\sin\theta\frac{dr}{dt}\cdot\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2},$$

$$\frac{d^2y}{dt^2} = \sin\theta\frac{d^2r}{dt^2} + 2\cos\theta\frac{dr}{dt}\cdot\frac{d\theta}{dt} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}.$$

Hence substituting

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P,$$

$$2\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\cdot\frac{d^2\theta}{dt^2} = T.$$

These are the polar equations of motion.

The preceding transformation is the most direct, and leaves the problem still in the form of two differential equations of the second order between r, θ and t.

The following transformation is more useful though more indirect: we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -P\cos\theta - T\sin\theta\\ \frac{d^2y}{dt^2} &= -P\sin\theta + T\cos\theta\end{aligned}\right\} \ldots\ldots\ldots (1).$$

From these

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = Tr, \ldots\ldots\ldots\ldots (2)$$

Let $x\dfrac{dy}{dt} - y\dfrac{dx}{dt} = h$;

then, $\dfrac{dh}{dt} = Tr$;

and since $r^2\dfrac{d\theta}{dt} = x\dfrac{dy}{dt} - y\dfrac{dx}{dt} = h$,

$$\frac{dh}{dt} = \frac{dh}{d\theta}\cdot\frac{d\theta}{dt} = \frac{h}{r^2}\cdot\frac{dh}{d\theta};$$

$$\therefore\ h\frac{dh}{d\theta} = Tr^3 = \frac{T}{u^3},$$

and integrating

$$h^2 = h_1^2 + 2\int\frac{T}{u^3}d\theta, \ldots\ldots\ldots\ldots (3)$$

where $\dfrac{1}{u}$ is written for r.

Again, $\dfrac{dr}{dt} = -\dfrac{1}{u^2}\cdot\dfrac{du}{d\theta}\cdot\dfrac{d\theta}{dt} = -h\dfrac{du}{d\theta}$;

$$\therefore\ \frac{d^2r}{dt^2} = -\frac{dh}{dt}\cdot\frac{du}{d\theta} - h^2u^2\cdot\frac{d^2u}{d\theta^2}.$$

Substituting this value of $\dfrac{d^2r}{dt^2}$ in the equation

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 + P = 0,$$

we have $\dfrac{dh}{dt}\cdot\dfrac{du}{d\theta} + h^2u^2\cdot\dfrac{d^2u}{d\theta^2} + h^2u^3 - P = 0$;

and therefore, $\dfrac{d^2u}{d\theta^2} + u - \dfrac{P}{h^2u^2} + \dfrac{T}{h^2u^3}\cdot\dfrac{du}{d\theta} = 0$;

and substituting for h^2 the value given by (3) we obtain

$$\frac{d^2u}{d\theta^2} + u - \frac{\dfrac{P}{h_1^2u^2} - \dfrac{T}{h_1^2u^3}\cdot\dfrac{du}{d\theta}}{1 + 2\displaystyle\int\frac{T}{h_1^2u^3}d\theta} = 0 \ldots\ldots (4),$$

$$\text{and } \frac{d\theta}{dt} = \frac{h}{r^2} = h_1 u^2 \sqrt{1 + 2\int \frac{T}{h_1^2 u^2} d\theta} \ldots\ldots (5).$$

Equations (4) and (5) though very complicated in appearance will not be so in practice, since in the cases which occur the forms of P and T will be such as to introduce considerable simplifications.

When the whole force which acts on the body is central, we have $T = 0$, and these equations become much simpler. We have in this case

$$\frac{d^2 u}{d\theta^2} + u - \frac{P}{h_1^2 u^2} = 0,$$

$$\text{and } \frac{dt}{d\theta} = \frac{1}{h_1 u^2}.$$

These equations, however, and the class of problems depending on them deserve an independent consideration.

CHAPTER III.

54. WHEN the direction of the force that acts on a particle always passes through a fixed point the force is said to be central, and that point is called the center of force. In the cases that will present themselves the intensity of the force will generally be a function of the distance of the particle from the center and of nothing else. Many of the properties of motion round centers of force are true, whether this is or is not the case. The student will, however, see from the proof whether this is requisite or not.

We will first shew, that however a body be projected, if it is subject to the action of a central force only, it will always move in one plane: and we shall afterwards confine our attention to that plane. Let P be the accelerating-force to the center, x, y, z the co-ordinates of the particle at time t, r its distance from the origin which is the center of force. Then the equations of motion are

$$\frac{d^2x}{dt^2} = -\frac{Px}{r},$$

$$\frac{d^2y}{dt^2} = -\frac{Py}{r},$$

$$\frac{d^2z}{dt^2} = -\frac{Pz}{r}.$$

Combining these equations two and two, we obtain

$$y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2} = 0,$$

$$z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2} = 0,$$

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0.$$

Integrating these, we have

$$y\frac{dz}{dt} - z\frac{dy}{dt} = h_1,$$

$$z\frac{dx}{dt} - x\frac{dz}{dt} = h_2,$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h_3,$$

where h_1, h_2, h_3 are constants introduced in integration. Multiplying them by x, y, and z respectively and adding, we get

$$0 = h_1 x + h_2 y + h_3 z,$$

a relation to which x, y, and z are always subject.

But this is the equation to a plane passing through the origin, therefore the body is at every instant in the same plane, and this is its equation.

The curved line in which a body moves round a center of force is called its orbit.

It may be seen from the above equations that the areas described on the co-ordinate planes by the projections of the radius vector are proportional to the times of describing them.

We will for the future consider the plane of motion as that of xy, and retain the rest of the notation.

55. The equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -P\frac{x}{r} = -P\cos\theta \\ \frac{d^2y}{dt^2} &= -P\frac{y}{r} = -P\sin\theta \end{aligned}\right\}, \quad \ldots\ldots\ldots(1)$$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \ldots\ldots\ldots(2)$$

$$x^2 + y^2 = r^2, \quad \ldots\ldots\ldots(3)$$

as before, $x\dfrac{dy}{dt} - y\dfrac{dx}{dt} = h.$

Now $x\dfrac{dy}{dt} - y\dfrac{dx}{dt} = r^2\dfrac{d\theta}{dt}$;

$$\therefore\ r^2\frac{d\theta}{dt} = h \quad \ldots\ldots\ldots(4).$$

It is found convenient to use the reciprocal of r instead of r, and to denote it by u, so that

$$\frac{d\theta}{dt} = hu^2 \quad \ldots\ldots\ldots(5).$$

Now since $x = \dfrac{1}{u}\cos\theta$;

$$\therefore\ \frac{dx}{dt} = -\frac{\sin\theta}{u}\frac{d\theta}{dt} - \frac{\cos\theta}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -hu\sin\theta - h\cos\theta\frac{du}{d\theta};$$

$$\therefore \frac{d^2x}{dt^2} = -hu\cos\theta\frac{d\theta}{dt} - h\cos\theta\frac{d^2u}{d\theta^2}\frac{d\theta}{dt}$$

$$= -h^2u^3\cos\theta - h^2u^2\cos\theta\frac{d^2u}{d\theta^2}.$$

Substituting this in the first of equations (1), and dividing by $\cos\theta$, we have

$$h^2u^3 + h^2u^2\frac{d^2u}{d\theta^2} = P,$$

$$\text{or } \frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} = 0 \ldots\ldots\ldots (6).$$

Equations (5) and (6) are the polar equations of motion: they are equivalent to the two equations (1).

Since however equations (1) are both of the second order, their integrals will contain four arbitrary constants. Equation (5) is of the first order, consequently only three constants will be introduced in the solution of equations (5) and (6). The remaining one h has been already introduced in the process of transformation.

These two may be considered as fundamental equations by which all questions relating to central forces may be solved. The constants which are introduced in integration are determined by various conditions which form part of the data of the problem. The quantity h which occurs in these equations is twice the sectorial area swept out by the radius vector in a unit of time. It is constant throughout the motion, and must be known in order that the problem may be completely solved. It will therefore be found useful to express it in terms of some of the other quantities that enter our equations.

We have $h = r^2\frac{d\theta}{dt}$

$$= \frac{1}{u^2}\frac{d\theta}{dt}.$$

Now $v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$

$$= \left(\frac{ds}{dt}\right)^2$$

$$= r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2$$

$$= \frac{1}{u^4}\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\}\left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{1}{p^2u^4}\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{u^2\sin^2\phi}\left(\frac{d\theta}{dt}\right)^2.$$

Where p and ϕ are, as usual, the perpendicular on the tangent, and the angle between the tangent and radius vector.

From these equations we have the following relations:

$$\left.\begin{aligned} h^2 &= p^2v^2 \\ h^2 &= \frac{v^2\sin^2\phi}{u^2} = v^2r^2\sin^4\phi \\ v^2 &= h^2\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots(7).$$

These relations are useful in solving the equations

$$hu^2 = \frac{d\theta}{dt}, \quad \ldots\ldots\ldots\ldots\ldots(5)$$

$$\frac{d^2u}{d\theta^2}+u-\frac{P}{h^2u^2}=0 \quad \ldots\ldots\ldots\ldots(6).$$

Though equations (5) and (6) are generally preferable, as being the most direct, and giving the solutions under the form of an equation between r and θ, there are some other relations which in particular cases are advantageous. We have

$$\frac{d^2x}{dt^2} = -P\frac{x}{r},$$

$$\frac{d^2y}{dt^2} = -P\frac{y}{r};$$

$$\therefore \frac{dx}{dt}\frac{d^2x}{dt^2}+\frac{dy}{dt}\frac{d^2y}{dt^2} = -P\cdot\frac{1}{r}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right),$$

$$\text{or } v\frac{dv}{dt} = -P\frac{dr}{dt} \quad \ldots\ldots\ldots\ldots(8).$$

From this equation the velocity can at once be determined as a function of the distance. We have by integrating it

$$v^2 = f(r)+C.$$

If V be the velocity at distance R we have

$$V^2 = f(R)+C;$$

$$\therefore v^2-V^2 = f(r)-f(R).$$

This equation is a particular case of the more general one obtained in Art. 50.

Equation (8) may be written

$$\frac{d.v^2}{dr} = -2P,$$

and is easily remembered under that form.

Again, from (7) we have

$$v = \frac{h}{p};$$

$$\therefore \log v = \log h - \log p,$$

and differentiating $\frac{1}{v} \cdot \frac{dv}{dt} = -\frac{1}{p} \cdot \frac{dp}{dt}$;

and from (8) $v \frac{dv}{dt} = -P \frac{dr}{dt}$;

$$\therefore v^2 = P \cdot p \frac{dr}{dp} \ldots\ldots (9).$$

If q be the chord of curvature of the orbit through the center of force

$$q = 2p \frac{dr}{dp};$$

therefore equation (9) may be written

$$v^2 = 2P \cdot \frac{q}{4} \quad \ldots\ldots (10)$$

which shews that the velocity at any point is that which would be acquired in falling through a distance equal to a quarter of the chord of curvature of the orbit at that point, under the action of a constant force equal in intensity to the force at that point.

Again, combining the equation

$$v^2 = \frac{h^2}{p^2}$$

with equation (9) we have

$$P = \frac{h^2}{p^3} \frac{dp}{dr}, \quad \ldots\ldots (11)$$

another relation which is sometimes found useful.

The results here obtained agree with those arrived at by independent considerations in *The Principia*, Book I. Section 2, with which they should be carefully compared.

56. We will now apply these equations to a very important example; namely, where the force varies inversely as the square of the distance.

This, it will be remembered, is the law of attraction in nature, and it is from this fact that it derives its importance.

We have in this case $P = \mu u^2$,

and substituting this in the equation

$$\frac{d^2u}{d\theta^2} + u - \frac{P}{h^2 u^2} = 0,$$

we get the equation

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2} = 0 \ldots\ldots (1).$$

The integral of this equation is

$$u = \frac{\mu}{h^2}\{1 + e\cos(\theta - \alpha)\} \ldots\ldots\ldots (2).$$

This is well known as the polar equation to a conic section, the focus being the pole, and $\theta - \alpha$ the angular distance from the nearer vertex.

Hence then a body moving about a center of force which varies inversely as the square of the distance will move in some conic section.

The constant quantities e and α which have been introduced in integration must be determined from the circumstances of the motion at some known epoch, or other equivalent data; h also must be determined from similar considerations. For example, suppose it known that, when $\theta = 0$, $u = c_1$, the velocity $= V$ and the direction of motion makes an angle β with the radius vector: differentiating equation (2) we get

$$\frac{du}{d\theta} = -\frac{\mu}{h^2} \cdot e\sin(\theta - \alpha);$$

$$\text{and } \therefore \cot\phi = -\frac{1}{u} \cdot \frac{du}{d\theta}$$

$$= \frac{1}{u} \cdot \frac{\mu}{h^2} \cdot e\sin(\theta - \alpha).$$

Hence we get $c_1 = \frac{\mu}{h^2} \cdot (1 + e\cos\alpha)$,

$$\cot\beta = -\frac{1}{c_1} \cdot \frac{\mu}{h^2} \cdot e\sin\alpha,$$

$$h^2 = \frac{V^2}{c_1^2} \cdot \sin^2\beta.$$

The last equation determines h, the other equations give

$$e\cos\alpha = \frac{V^2\sin^2\beta}{\mu c_1} - 1,$$

$$e\sin\alpha = -\frac{V^2}{\mu c_1} \cdot \sin\beta\cos\beta;$$

$$\therefore e^2 = \frac{V^4\sin^2\beta}{\mu^2 c_1^2} - 2 \cdot \frac{V^2\sin^2\beta}{\mu c_1} + 1,$$

which gives the magnitude of e, and it was assumed to be positive

when $\theta - \alpha$ was considered as measured from the nearer vertex: α may then be found without ambiguity from the two preceding equations.

The constants which enter into equation (2) are then determined and the equation to the orbit is completely known.

If we had not substituted for h^2 we should then have had

$$e \cos \alpha = \frac{c_1 h^2}{\mu} - 1,$$

$$e \sin \alpha = -\frac{c_1 h^2}{\mu} . \cot \beta ;$$

$$\text{and } \therefore \ e^2 = \frac{c_1^2 h^4}{\mu^2} . \operatorname{cosec}^2 \beta - 2 . \frac{c_1 h^2}{\mu} + 1,$$

which is convenient when h and not V is known. V, c_1, β are called the circumstances of projection.

The nature of the conic section will of course depend on the magnitude of e. If $e = 0$, the orbit is a circle. That this may be the case we must have

$$\frac{V^4 \sin^2 \beta}{\mu^2 c_1^2} - 2 . \frac{V^2 \sin^2 \beta}{\mu c_1} + 1 = 0.$$

To satisfy this equation we must have $\beta = \frac{\pi}{2}$, (a condition which is also evident independently) and $V^2 = \mu c_1$.

This condition agrees with that deduced previously for motion in a circle.

The velocity of a body moving in a circle is often taken as a standard by comparison with which the velocities of bodies in their orbits are estimated. Thus the velocity of a body is sometimes expressed by saying that it is q times that in a circle at the same distance. The substitution of $q^2 \mu c_1$ for V^2 will simplify the equations. It gives

$$e^2 = q^4 \sin^2 \beta - 2q^2 \sin^2 \beta + 1.$$

The most important case is where e is less than unity, or the orbit is elliptical.

$$\text{Now } e^2 = (q^2 - 1)^2 \sin^2 \beta + \cos^2 \beta.$$

Hence that e may be less than unity q^2 must be less than 2, or the velocity at every point is less than $\sqrt{2} \times$ that in a circle at the same distance.

Let a be the semi-axis major of the ellipse; then from equation (2)

$$\frac{1}{a} = \frac{\mu}{h^2}(1-e^2)$$

$$= \frac{\mu c_1^2}{V^2 \sin^2\beta}\left(\frac{2V^2\sin^2\beta}{\mu c_1} - \frac{V^4\sin^2\beta}{\mu^2 c_1^2}\right)$$

$$= 2c_1 - \frac{V^2}{\mu};$$

an important equation, which shews that the axis-major of the orbit is independent of the direction of projection, and depends only on the distance and velocity.

If we write v for V, and $\frac{1}{r}$ for c_1, we have,

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right),$$

which gives the velocity at any distance in a simple and useful form.

a determines the position of the axis-major of the orbit: when the body is at the extremity of the axis-major its motion is perpendicular to the radius-vector: all places in any orbit where the direction of the motion of the body is perpendicular to the radius-vector are called "apsides," or, more commonly, "apses." α is the angular distance of the apse from the line corresponding to $\theta = 0$, or, as it is commonly called, the "longitude" of the apse.

θ is the "longitude" of the body; e, the eccentricity, determines the form of the orbit, and a, the semi-axis major; determines its magnitude.

α, e and a are called the "elements" of the orbit; when they are known, the form, magnitude, and position of the orbit are completely determined.

The first object of the direct problem, therefore, is to determine the elements of the orbit from the circumstances of projection, or other equivalent conditions. Sometimes it is required to determine the circumstances of projection from certain given conditions which the motion must satisfy.

We have seen that for the orbit to be an ellipse we must have e less than unity, which requires that q be less than $\sqrt{2}$, or V^2 less than $2\mu c_1$.

When $q = \sqrt{2}$,

or $V^2 = 2\mu c_1$,

we have $e = 1$, and the orbit is a parabola. The equation then becomes

$$u = \frac{\mu}{h^2}\{1 + \cos(\theta - \alpha)\},$$

$$\frac{h^2}{\mu} = \frac{V^2 \sin^2\beta}{\mu c_1^2} = \frac{2\sin^2\beta}{c_1}$$

is the semi-latus-rectum of the parabola. α, as in the previous case, is the longitude of the apse. When these are known the orbit is completely determined.

The velocity at any distance r in a parabolic orbit is given by the very simple equation

$$v^2 = \frac{2\mu}{r}.$$

If q be greater than $\sqrt{2}$, or V greater than $\sqrt{2\mu c_1}$, we shall have e greater than unity, and the orbit will be an hyperbola. The elements in this case will be determined by a process similar to that used in the ellipse. This case, however, is of comparatively small importance.

We have at present only found the equation to the orbit, and its nature, form, magnitude and position; and these, we have seen, remain constant. In order that the problem may be completely solved, we must moreover determine the position in the orbit which the body occupies at any instant: that is, we must find u and θ, or r and θ as functions of t.

To do this we must integrate the equation

$$\frac{d\theta}{dt} = hu^2,$$

having given $u = \frac{\mu}{h^2}\{1 + e\cos(\theta - \alpha)\}$,

from which $\frac{d\theta}{dt} = \frac{\mu^2}{h^3}\{1 + e\cos(\theta - \alpha)\}^2$.

This could be integrated accurately in finite terms, the solution, however, which is thus obtained is in a form which is practically useless; as, however, in the cases in which it is generally required e is a small quantity, the solution is obtained in the form of a series ascending by powers of e, and may be carried to any degree of accuracy.

We are able, without integrating the equation at all, to determine the time of a complete revolution. For since h is twice the area

swept out in the unit of time, if we divide twice the whole area of the ellipse by h the quotient will be the time of a complete revolution. Calling this T, we have

$$T = \frac{2\pi a^2 \sqrt{1-e^2}}{h},$$

or since $\frac{\mu}{h^2} = \frac{1}{a(1-e^2)}$,

$$T = 2\pi \sqrt{\frac{a^3}{\mu}}.$$

T is called the "period" or "periodic time" of the body.

In the time T the longitude of the body increases by 2π, but this increase does not take place uniformly. If it did take place uniformly, the increase of longitude in time t would be

$$\frac{t}{T} \cdot 2\pi,$$

$$\text{which} = t \cdot \sqrt{\frac{\mu}{a^3}}.$$

n is generally used for the quantity $\sqrt{\frac{\mu}{a^3}}$, and therefore the increase of longitude in time t on this false supposition is nt; n is called the "mean motion" of the body; it is useful in simplifying some of the equations.

It is connected with the other constants that have been used by the equations

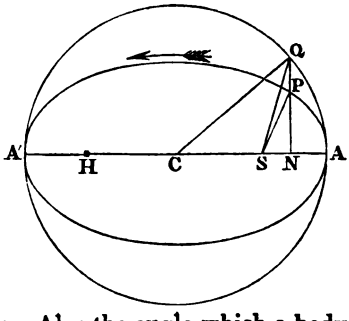
$$n^2 = \frac{\mu}{a^3},$$

$$h^2 = n^2 a^4 (1-e^2).$$

We will now proceed to determine θ as a function of t. To facilitate this certain auxiliary quantities are used which we will now define.

In Astronomy (on which this part of the subject more particularly bears) the name "anomaly" is given to certain angles that are frequently used.

Let APA' be the ellipse which the body describes; AQA' a circle on the axis-major; S the focus coinciding with the center of force; P the position of the particle at the time t.



Then A is the nearer apse and A' is the farther apse. The angle ASP which is what has been written $\theta - \alpha$ is the angular distance from the apse. It is called the "true anomaly." We will denote it by the letter v.

If the ordinate NP be produced to meet the circle in Q, the angle ACQ is called the "eccentric anomaly," and is denoted by the letter u. Also the angle which a body moving with the mean motion n would describe while the real body is moving from A to P, is called the "mean anomaly," and is denoted by the letter m.

We have then $\dfrac{m}{2\pi} = \dfrac{\text{area } ASP}{\text{area of ellipse}} = \dfrac{\text{area } ASQ}{\pi a^2}$.

$$\text{Now area } ASQ = \text{area } ACQ - \text{area } SCQ$$

$$= \tfrac{1}{2} a^2 u - \tfrac{1}{2} ae \,.\, a \sin u;$$

$$\therefore\ m = u - e \sin u;$$

$$\text{also, } r = a - e \,.\, CN$$

$$= a(1 - e \cos u);$$

$$\therefore\ \cos v = \frac{SN}{SP} = \frac{a(\cos u - e)}{a(1 - e\cos u)} = \frac{\cos u - e}{1 - e \cos u};$$

$$\therefore\ \frac{1 - \cos v}{1 + \cos v} = \frac{1 + e}{1 - e} \cdot \frac{1 - \cos u}{1 + \cos u},$$

$$\therefore\ \tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \,.\, \tan \frac{u}{2}.$$

$$\text{Now } v = \theta - a;$$

$$\therefore\ \frac{dv}{dt} = \frac{d\theta}{dt} = \frac{h}{r^2}$$

$$= \frac{h}{a^2(1 - e \cos u)^2}$$

$$= n(1 - e^2)^{\frac{1}{2}} (1 - e \cos u)^{-2}$$

$$= n\left(1 - \frac{e^2}{2} - \frac{e^4}{2^3} \ldots\right)(1 + 2e \cos u + 3e^2 \cos^2 u + 4e^3 \cos^3 u + \&c.)$$

$$= n\left\{1 + 2e \cos u + \frac{e^2}{2}(6 \cos^2 u - 1) + \&c.\right\}.$$

From the equation $m = u - e \sin u$

we must express $\cos u$, $\cos^2 u$, &c. as functions of m by means of Lagrange's theorem, and transform them so as to involve only simple powers of cosines of multiples of m.

$$\text{Now } \frac{dm}{dt} = n;$$

$$\therefore\ m = nt + C.$$

We now have $\frac{dv}{dt}$ expressed in a series of simple powers of cosines of $nt + C$ and its multiples, and can therefore find v at once by integration.

If we only proceed as far as e^2 it is not necessary to employ Lagrange's theorem. We have

$$u = m + e \sin u.$$

To a first approximation, $u = m$, where the error committed is of the same order of magnitude as e.

Therefore to a second approximation

$$u = m + e \sin m;$$

$$\begin{aligned}\therefore\ \cos u &= \cos (m + e \sin m)\\ &= \cos m \,.\, \cos (e \sin m) - \sin m \,.\, \sin (e \sin m)\\ &= \cos m - e \sin^2 m\\ &= \cos m - \frac{e}{2}(1 - \cos 2m),\end{aligned}$$

the error being of the order e^2.

$$\text{Also, } \cos^2 u = \frac{1 + \cos 2u}{2}$$

$$= \frac{1 + \cos 2m}{2},$$

with an error of the order e.

Therefore, substituting, we have

$$\begin{aligned}\frac{dv}{dt} &= n\left\{1 + 2e\cos m + \frac{5}{2}e^2 \cos 2m\right\}\\ &= n\left\{1 + 2e\cos (nt + C) + \frac{5}{2}e^2 \cos 2(nt + C)\right\},\end{aligned}$$

where the error is of the order e^3;

$$\therefore v = nt + C' + 2e \sin (nt + C) + \frac{5}{4}e^2 \sin 2(nt + C).$$

Now $v = 0$ when $m = 0$, that is, when $nt + C = 0$;

$$\therefore\ C' = C;$$

$$\therefore\ \theta - \alpha = v = nt + C + 2e\sin(nt + C) + \frac{5}{4}e^2\sin 2(nt + C).$$

It is customary to use ϖ instead of α, and instead of C to write $\epsilon - \varpi$, so that

$$\theta = nt + \epsilon + 2e\sin(nt + \epsilon - \varpi) + \frac{5e^2}{4}\sin 2(nt + \epsilon - \varpi);$$

θ is the true longitude of the body; we observe that it consists of a part $nt + \epsilon$ which increases uniformly, together with a number of terms which are sometimes positive and sometimes negative, but which always lie between certain limits; the periods in which they go through all their values being different.

$nt + \epsilon$ is called the "mean longitude."

ϵ is the mean longitude when $t = 0$, it is called the "epoch," it is the new constant that has been introduced in the integration; it is of course known when the longitude of the body at any one instant is known.

ϖ is the longitude of the apse, $nt + \epsilon - \varpi$ is the mean anomaly.

$v - m$ or $\theta - (nt + \epsilon)$ is sometimes called the "equation to the center," and sometimes the "elliptic inequality."

It still remains to express r the distance of the body from the center as a function of t.

This is easily done; we have

$$r = a(1 - e\cos u).$$

Substituting for $\cos u$ its approximate value, we have

$$r = a\left\{1 - e\cos m + \frac{e^2}{2}(1 - \cos 2m)\right\}$$

$$= a\left\{1 - e\cos(nt + \epsilon - \varpi) + \frac{e^2}{2}[1 - \cos 2(nt + \epsilon - \varpi)]\right\},$$

where the error is of the order e^3.

We have thus obtained an approximation to the complete solution of the problem.

By using Lagrange's Theorem, we might have carried this approximation to any degree of accuracy.

The four constants introduced in integration which constitute the four elements of the orbit are ϖ, e, a, and ϵ.

The first determines the position of the orbit, the second its form, the third its magnitude, and the fourth the position of the body in it.

When a and e are known, n and h are also known.

In solving the problem, we have considered μ to be a known quantity, when this is not the case it is usual to consider n as the quantity to be determined, since when n and a are both known, μ is known from the equation $\mu = n^2 a^3$.

The conditions for determining these elements will appear under very different forms. One very important case is, when we know the values of θ corresponding to four different values of t, that is, the longitude at four different times.

These will give us four equations,

$$\theta_1 = nt_1 + \epsilon + 2e \sin(nt_1 + \epsilon - \varpi) + \&c.,$$

$$\theta_2 = nt_2 + \epsilon + \&c.,$$

$$\theta_3 = nt_3 + \epsilon + \&c,$$

$$\theta_4 = nt_4 + \epsilon + \&c.$$

These equations enable us to determine e, n, ϵ, and ϖ; and if μ is known, we then know a. If μ is not known we may remark that more values of θ will not assist in determining it, since the above four quantities are the only ones which enter into the equations.

If we had the four longitudes, and the distance or value of r at any given instant we could also find a and thence μ.

Or we might have three longitudes and two distances, or these conditions might be varied in any way.

It is however necessary to have one distance at least, and one longitude at least in order to determine a and ϵ.

If μ is given the distance is not necessary.

The method of approximation which we adopted in obtaining the above equations is one with which the student will do well to make himself familiar as early as possible. It will be continually occurring in the higher parts of physical and plane Astronomy, the same in principle but with variations in the form of applying it.

57. There is also another method which is largely used in the Planetary Theory, which we will illustrate by the problem which is now before us. Let us first, however, collect the principal equations which connect the elements of the orbit and the circumstances of projection.

$$e^2 = \frac{V^4 \sin^2\beta}{\mu^2 c_1^2} - 2\frac{V^2 \sin^2\beta}{\mu c_1} + 1 \ldots\ldots\ldots (1)$$

$$= \frac{c_1^2 h^4}{\mu^2} \operatorname{cosec}^2\beta - 2\frac{c_1 h_2}{\mu} + 1 \ldots\ldots\ldots (2)$$

$$h = \frac{V}{c_1} \sin\beta \ldots\ldots\ldots (3)$$

$$\tan\alpha = \frac{V^2 \sin\beta\cos\beta}{\mu c_1 - V^2 \sin^2\beta} \ldots\ldots\ldots (4)$$

$$a(1-e^2) = \frac{h^2}{\mu} \ldots\ldots\ldots (5)$$

$$V^2 = \mu\left(2c_1 - \frac{1}{a}\right) \ldots\ldots\ldots (6)$$

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \ldots\ldots\ldots (7)$$

$$n^2 = \frac{\mu}{a^3} \ldots\ldots\ldots (8)$$

The question which we propose to consider is the nature and magnitude of the change which would be produced in the elements of the orbit by a small change in the circumstances of projection, or of the intensity of the force, or the converse.

The manner of doing this will be best shewn by taking two or three particular cases.

Suppose that the velocity of projection is slightly increased, the distance and direction remaining unaltered. We see, by inspecting the preceding equations, that any change in V will be accompanied by a change in most or all of the elements, and consequently that these elements are functions of V.

If therefore we suppose V to be increased to $V + \delta V$, we must substitute $V + \delta V$ for V in the above equations, in order to obtain the new values of the elements. We may expand these new values by Taylor's Theorem in a series ascending by powers of δV: and if, as we suppose, δV is small, we shall obtain these new values very approximately by neglecting all the terms of the series which contain powers of δV higher than the first; so that the new value of a will be very approximately

$$a + \frac{da}{dV} \cdot \delta V.$$

From equation (1), we have

$$e \cdot \delta e = 2\frac{V \sin^2\beta}{\mu c_1} \cdot \left(\frac{V^2}{\mu c_1} - 1\right) \cdot \delta V.$$

This shews that an increase of the velocity at any point would increase or decrease the eccentricity according as V^2 is greater or less than μc; that is, according as the velocity is greater or less than that in a circle at the same distance. From equation (6) we see that V^2 is greater or less than μc_1, as c_1 is greater or less than $\frac{1}{a}$; that is, as the distance is less or greater than a. Now the distance is a when the body is at the extremity of the axis-minor. Hence then an increase of the velocity will be accompanied by an increase or decrease of the eccentricity according as the body is at the time on the nearer or farther side of the axis-minor. Exactly opposite results would follow a diminution of the velocity.

Again, from equation (6)

$$\frac{1}{a} = 2c_1 - \frac{V^2}{\mu};$$

$$\therefore \frac{1}{a^2} . \delta a = \frac{2V}{\mu} . \delta V,$$

which shews that an increase of the velocity is always accompanied by an increase of the axis-major.

From equation (4) we have

$$\sec^2 a . \delta a = \frac{2\mu c_1 V \sin\beta \cos\beta}{(\mu c_1 - V^2 \sin^2\beta)^2} . \delta V.$$

Since $\cos\beta$ is the only factor of this expression which can change its sign, we see that an increase of the velocity will cause an increase or decrease of the longitude of the apse according as β is less or greater than $\frac{\pi}{2}$; that is, according as the body is moving from the nearer apse to the farther, or from the farther to the nearer.

As another example suppose V to remain unaltered, and the angle β to be slightly increased.

We have from equation (1)

$$e . \delta e = \frac{V^2}{\mu c_1} . \left(\frac{V^2}{\mu c_1} - 2 \right) . \sin\beta . \cos\beta . \delta\beta.$$

Here, since V^2 is less than $2\mu c_1$, e will be increased or diminished according as $\cos\beta$ is negative or positive, that is, as β is greater or less than $\frac{\pi}{2}$. An exactly opposite result would follow a diminution of β; so that we see in all cases e is increased or diminished as β is made farther from, or nearer to, a right angle.

From equation (6) we see that a is unaffected by a change in β; a result which we have already remarked.

From equation (4) we get

$$\sec^2 a\, \delta a = V^2 \frac{\mu c_1 \cos^2\beta + (V^2 - \mu c_1)\sin^2\beta}{(\mu c_1 - V^2 \sin^2\beta)^2}\, \delta\beta,$$

which shews that the longitude of the apse will be increased or decreased by an increase of β, according as

$$\mu c_1 \cos^2\beta + (V^2 - \mu c_1)\sin^2\beta,$$

or, $\mu(ac_1 - \sin^2\beta)$ is positive or negative.

When the body is on the nearer side of the axis-minor ac_1 is greater than unity, and therefore this is positive; when the body is very near the farther apse $ac_1 - \sin^2\beta$ is negative; the determination of the point where $(ac_1 - \sin^2\beta)$ changes sign is a geometrical problem of little importance. We see, therefore, that by an increase of β which takes place near the nearer apse the longitude of the apse is increased, and diminished by an increase of β which takes place near the farther apse.

These will be sufficient to illustrate the method, which goes by the name of the "Variation of Parameters."

We have selected these examples as themselves of importance. The results should be carefully compared with the Corollaries to the 66th Proposition of the eleventh Section of the *Principia*.

We have dwelt at considerable length on the motion in an ellipse because it forms the basis of the calculations of the motions of the heavenly bodies.

58. These methods of expressing θ and r as functions of t entirely fail when the motion is in a parabola or hyperbola, since the convergence of the series depends on e being less than unity. In the case of a parabola, however, the equation can be easily integrated.

$$\frac{dt}{d\theta} = \frac{r^2}{h} = \frac{h^3}{\mu^3} \cdot \frac{1}{\{1 + \cos(\theta - \alpha)\}^2}$$

$$= \frac{h^3}{4\mu^3} \cdot \sec^4 \frac{\theta - \alpha}{2}.$$

Let $2l$ be the latus-rectum of the parabola, then $h^2 = \mu l$,

$$\text{and } \frac{dt}{d\theta} = \frac{1}{4}\sqrt{\frac{l^3}{\mu}} \cdot \left(1 + \tan^2\frac{\theta - \alpha}{2}\right)\sec^2\frac{\theta - \alpha}{2};$$

$$\therefore\ t + C = \frac{1}{2}\sqrt{\frac{l^3}{\mu}} \cdot \left(\tan\frac{\theta - \alpha}{2} + \frac{1}{3}\tan^3\frac{\theta - \alpha}{2}\right).$$

From this equation θ may be expressed in terms of t.

59. The method of integrating the equation

$$\frac{d^2u}{d\theta^2}+u-\frac{P}{h^2u^2}=0,$$

which has been adopted in the case of

$$P=\mu u^3,$$

is not generally applicable.

Suppose $P=\mu u^n$; we must proceed as follows:

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}\cdot u^{n-2}\ldots\ldots(1)$$

Multiplying both sides by $2\frac{du}{d\theta}$, and integrating, we have

$$h^2\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=\frac{2\mu}{n-1}u^{n-1}+C.$$

Let V be the velocity of the particle when $u=c$, then

$$V^2=\frac{2\mu}{n-1}\cdot c^{n-1}+C;$$

$$\therefore\ h^2\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=V^2+\frac{2\mu}{n-1}(u^{n-1}-c^{n-1}).$$

It will be convenient to express V in terms of the velocity which a body would acquire in moving from rest at an infinite distance to a distance $\frac{1}{c}$ from the center of force: to find this we have

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^n},$$

$$\left(\frac{dx}{dt}\right)^2=C+\frac{2}{n-1}\cdot\frac{\mu}{x^{n-1}},$$

$$0=C;$$

$$\therefore\ V_1^2=\frac{2\mu}{n-1}\cdot c^{n-1}.$$

Let $V=qV_1$, then

$$h^2\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=\frac{2\mu}{n-1}\{u^{n-1}+(q^2-1)c^{n-1}\};$$

$$\therefore\ \frac{d\theta}{du}=\frac{h\sqrt{n-1}}{\sqrt{2\mu u^{n-1}-(n-1)h^2u^2+2\mu(q^2-1)c^{n-1}}}\ldots\ldots(2)$$

This equation can always be integrated when $q=1$. In this case it becomes

$$\frac{d\theta}{du}=\frac{h\sqrt{n-1}}{u\sqrt{2\mu u^{n-3}-(n-1)h^2}}.$$

Let $\frac{2\mu u^{n-3}}{(n-1)h^2}=y^2$;

$$\text{then } \frac{d\theta}{dy} = \frac{2}{n-3} \frac{1}{y\sqrt{y^2-1}};$$

$$\therefore \theta - \alpha = \frac{2}{n-3} \sec^{-1} y;$$

$$\therefore \frac{2\mu}{(n-1)h^2} u^{n-3} = \sec^2 \frac{n-3}{2}(\theta - \alpha); \ldots\ldots (3)$$

$$\therefore r^{n-3} = \frac{2\mu}{(n-1)h^2} \cdot \cos^2 \frac{n-3}{2}(\theta - \alpha),$$

the general equation to the orbit described.

This method of solution fails when $n = 3$, and when $n = 1$.

When $n = 3$ equation (2) becomes

$$\frac{d\theta}{du} = \frac{1}{\sqrt{\left(\frac{\mu}{h^2} - 1\right)u^2 + \frac{\mu}{h^2}(q^2-1)c^2}}.$$

The integral of this equation will be logarithmic, algebraical, or trigonometrical as $\frac{\mu}{h^2}$ is greater than, equal to, or less than unity.

In the first case,

$$\theta - \alpha = \frac{1}{\sqrt{\frac{\mu}{h^2} - 1}} \log \left\{ u\sqrt{\frac{\mu}{h^2} - 1} + \sqrt{\left(\frac{\mu}{h^2} - 1\right)u^2 + \frac{\mu}{h^2}(q^2-1)c^2} \right\};$$

$$\therefore u\sqrt{\mu - h^2} + \sqrt{(\mu - h^2)u^2 + \mu(q^2-1)c^2} = h \cdot \epsilon^{(\theta-\alpha)\sqrt{\frac{\mu}{h^2}-1}}.$$

In this case the value of q can be either greater or less than unity.

In the second case we have $\frac{\mu}{h^2} = 1$, and the integral is

$$\theta - \alpha = \frac{u}{c} \frac{1}{\sqrt{q^2-1}},$$

$$\text{or } u = c\sqrt{q^2-1} \cdot (\theta - \alpha),$$

in this case q^2 can never be less than unity, for

$$q^2 = \frac{V^2}{\mu c^2} = \frac{h^2}{\mu \sin^2\beta} = \frac{1}{\sin^2\beta}.$$

Lastly, when $\frac{\mu}{h^2}$ is less than unity, we have

$$\frac{d\theta}{du} = \frac{1}{\sqrt{\frac{\mu}{h^2}(q^2-1)c^2 - \left(1 - \frac{\mu}{h^2}\right)u^2}};$$

$$\therefore \theta - \alpha = \frac{h}{\sqrt{h^2 - \mu}} \sin^{-1} \sqrt{\frac{h^2-\mu}{\mu(q^2-1)}} \cdot \frac{u}{c};$$

$$\therefore\ u = c\sqrt{\frac{\mu(q^2-1)}{h^2-\mu}}\,.\,\sin\sqrt{1-\frac{\mu}{h^2}}\,.(\theta-\alpha);$$

in this case also $q^2 = \dfrac{h^2}{\mu\sin^2\beta}$ is necessarily greater than unity.

We might have obtained these equations at once from the general equation

$$\frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} = 0,$$

which in this case becomes a linear equation

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{\mu}{h^2}\right)u = 0.$$

When $n = 1$, we have

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2u} = 0;$$

$$\therefore\ h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = 2\mu\log u + C,$$

$$V^2 = 2\mu\log c + C;$$

$$\therefore\ h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = 2\mu\log\frac{u}{c} + V^2.$$

We cannot generally perform the next integration. These cases however are of little importance. In all these cases after determining the equation to the orbit we must find the position of the body in the orbit from the equation $\dfrac{d\theta}{dt} = hu^2$.

60. It was stated in Article 56 that a point in an orbit where the motion of the body is at right angles to the radius-vector is called an apse, and the corresponding radius-vector is called the apsidal line, or apsidal distance. From the nature of an apse we must have at such points

$$\frac{dr}{d\theta} = 0;\ \text{and therefore}\ \frac{du}{d\theta} = 0.$$

Let $P = \mu u^n$, then the equation to the orbit is

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu u^{n-2}}{h^2} = 0;$$

$$\therefore\ \left(\frac{du}{d\theta}\right)^2 + u^2 - \frac{2\mu u^{n-1}}{(n-1)h^2} = C;$$

and at an apse we must have

$$u^2 - \frac{2\mu u^{n-1}}{(n-1)h^2} = C.$$

This equation cannot have more than two changes of sign, and therefore cannot have more than two positive roots: so that in an orbit round a center of force varying as any power of the distance there cannot be more than two different apsidal distances.

The angle between two consecutive apsidal distances is called the apsidal angle. When the orbit does not differ much from a circle this angle can be found approximately, without finding the equation to the orbit. Let $P = \mu u^2 \phi(u)$, then the differential equation of the orbit is

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2}\phi(u) = 0 \quad \ldots\ldots\ldots\ldots \quad (1)$$

Now since the orbit is nearly circular we have

$$u = c + x,$$

where x is small compared with u;

$$\therefore \ \phi(u) = \phi(c) + \phi'(c) \cdot x \text{ nearly},$$

$$\text{also } \frac{d^2u}{d\theta^2} = \frac{d^2x}{d\theta^2};$$

therefore equation (1) becomes

$$\frac{d^2x}{d\theta^2} + \{1 - \frac{\mu}{h^2} \cdot \phi'(c)\}x + c - \frac{\mu}{h^2} \cdot \phi(c) = 0 \quad \ldots(2)$$

also the velocity cannot differ much from that in a circle at the same distance, so that

$$\frac{\mu}{h^2} = \frac{c}{\phi(c)} \text{ nearly.}$$

Therefore the integral of equation (2) is

$$x = A \cos(k\theta + B) + C,$$

$$\text{where } k = \sqrt{1 - \frac{c \cdot \phi'(c)}{\phi(c)}} \text{ and } C = \frac{1}{k^2}\left\{\frac{\mu}{h^2}\phi(c) - c\right\};$$

$$\therefore \ \frac{dx}{d\theta} = -Ak \sin(k\theta + B).$$

Let θ_1, θ_2, θ_3, &c. be consecutive values of θ for which $\frac{dx}{d\theta} = 0$, then

$$k\theta_1 + B = n\pi,$$

$$k\theta_2 + B = (n+1)\pi,$$

$$k\theta_3 + B = (n+2)\pi,$$

$$\text{&c.};$$

$$\therefore \ \theta_3 - \theta_1 = \theta_3 - \theta_2 = \text{&c.} = \frac{\pi}{k},$$

which gives the apsidal angle approximately.

For example, let $P = \mu u^2 + \frac{\mu'}{u}$ where μ' is small: here

$$\phi(u) = 1 + \frac{\mu'}{\mu u^3};$$

$$\therefore\ \phi(c) = 1 + \frac{\mu'}{\mu c^3};$$

$$\phi'(c) = -\frac{3\mu'}{\mu c^4};$$

$$\therefore\ k^2 = 1 - \frac{c \cdot \phi'(c)}{\phi(c)} = \frac{\mu c^3 + 4\mu'}{\mu c^3 + \mu'};$$

$$\therefore \text{ the apsidal angle} = \pi \cdot \sqrt{\frac{\mu c^3 + \mu'}{\mu c^3 + 4\mu'}}.$$

61. The equations

$$P = h^2 u^2 \left(\frac{d^2 u}{d\theta^2} + u\right),$$

$$P = \frac{h^2}{p^3} \cdot \frac{dp}{dr},$$

enable us at once to solve the inverse problem of finding what must be the law of force in order that the particle may describe a given orbit. For example, let it be required to find the law of force that the orbit may be the equiangular spiral, whose equation is

$$r = a\epsilon^{\theta \cot \alpha},$$

$$\text{or } u = \frac{1}{a} \cdot \epsilon^{-\theta \cot \alpha};$$

$$\text{here } \frac{d^2 u}{d\theta^2} = \frac{1}{a} \cdot \cot^2 \alpha \cdot \epsilon^{-\theta \cot \alpha}$$

$$= u \cdot \cot^2 \alpha;$$

$$\therefore\ P = h^2 u^2 \left(\frac{d^2 u}{d\theta^2} + u\right)$$

$$= h^2 \operatorname{cosec}^2 \alpha \cdot u^3,$$

or thus, in the equiangular spiral,

$$p = r \sin \alpha;$$

$$\therefore\ \frac{dp}{dr} = \sin \alpha;$$

$$\therefore\ P = \frac{h^2}{p^3} \cdot \frac{dp}{dr} = \frac{h^2}{\sin^2 \alpha} \cdot \frac{1}{r^3}, \text{ as before.}$$

CHAPTER IV.

62. THE integration of the equations

$$M\frac{d^2x}{dt^2} = X,$$

$$M\frac{d^2y}{dt^2} = Y,$$

$$M\frac{d^2z}{dt^2} = Z,$$

in the cases where the motion is not, as we have hitherto supposed it, in one plane, is in general attended with great difficulties and can rarely be effected except approximately. As moreover no principles are involved in the solution of the equations in this case which have not been exemplified in the cases of motion in one plane, it is unnecessary to give any of the few instances in which the equations can be completely solved. There are however a few general properties of the motion which can be readily exhibited.

The resultant of the forces X, Y, Z acts in the osculating plane of the body's path.

This may be seen at once from general considerations.

The body at any instant is moving along the tangent to its path at that instant, and the deflection from the tangent must take place in the direction of the resultant force that acts on the particle at that point; but the plane passing through the tangent and the direction of the deflection from the tangent is the osculating plane, which therefore passes through the direction of the resultant force.

This may also be easily deduced from the equations.

The direction cosines of the normal to the osculating plane are proportional to

$$\frac{d^2y}{dt^2}\frac{dz}{dt} - \frac{d^2z}{dt^2}\frac{dy}{dt}, \quad \frac{d^2z}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dz}{dt} \text{ and } \frac{d^2x}{dt^2}\frac{dy}{dt} - \frac{d^2y}{dt^2}\frac{dx}{dt}.$$

Or, considering the equations of motion, to

$$Y\frac{dz}{dt} - Z\frac{dy}{dt}, \quad Z\frac{dx}{dt} - X\frac{dz}{dt}, \text{ and } X\frac{dy}{dt} - Y\frac{dx}{dt},$$

and the direction cosines of the resultant force at the point (x, y, z) are

$$\frac{X}{\sqrt{X^2+Y^2+Z^2}}, \quad \frac{Y}{\sqrt{X^2+Y^2+Z^2}} \text{ and } \frac{Z}{\sqrt{X^2+Y^2+Z^2}};$$

and if θ be the angle between the direction of the resultant force and the normal to the osculating plane, the numerator of $\cos\theta$ is zero, or θ is a right angle; that is, the resultant acts in the osculating plane.

Again, from the equations of motion we have

$$M\left(\frac{d^2x}{dt^2}\frac{dx}{dt}+\frac{d^2y}{dt^2}\frac{dy}{dt}+\frac{d^2z}{dt^2}\frac{dz}{dt}\right)=X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt},$$

$$\text{or since } \left(\frac{ds}{dt}\right)^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2,$$

$$M\frac{d^2s}{dt^2}=X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}.$$

The right hand side of this equation is the sum of the resolved parts of the forces X, Y, Z along the tangent, and this is the only part that affects the velocity: when this is zero, or the whole resultant force acts always in the normal plane, the velocity will be unaltered thoughout.

When the expressions for the forces X, Y, Z are such as to make $Xdx+Ydy+Zdz$ a perfect differential of x, y, z considered as three independent variables, that is, when they satisfy the conditions

$$\frac{dX}{dy}=\frac{dY}{dx}, \quad \frac{dY}{dz}=\frac{dZ}{dy}, \quad \frac{dX}{dz}=\frac{dZ}{dx},$$

$$\text{let } F(x, y, z)+C=2\int(Xdx+Ydy+Zdz),$$

$$\text{then } Mv^2=F(x, y, z)+C,$$

from which it appears that

$$F(x, y, z)=\text{const.}$$

represents a system of surfaces each of which has the property that the body always crosses it with the same velocity: and if any two of them be taken, as

$$F(x, y, z)=C_1,$$

$$F(x, y, z)=C_2,$$

the change of velocity in passing from one to the other is independent of the points in the two surfaces, and of the path pursued by the particle between them; for if v_1 and v_2 be the velocities at the two surfaces,

$$M v_1^2 = C_1 + C,$$
$$M v_2^2 = C_2 + C,$$
$$\text{so that } M(v_2^2 - v_1^2) = C_2 - C_1,$$

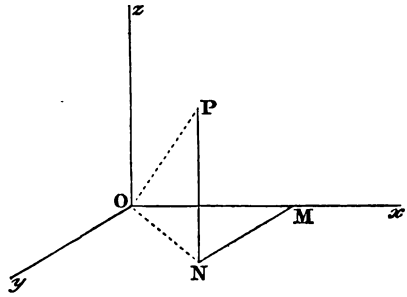
in which neither x, y, z nor any relation between x, y, and z, appear.

63. The three equations may be transformed to polar co-ordinates as follows:

Let P be the position of the particle,

$$\left.\begin{aligned} OM &= x, \\ MN &= y, \\ NP &= z. \end{aligned}\right\}$$

$$\left.\begin{aligned} OP &= r, \\ ON &= \rho, \\ MON &= \theta. \end{aligned}\right\}$$



Let P', T' and S' be the resolved parts of the forces on the particle in direction NO, perpendicular to NO, and in PN respectively,

$$\text{so that } M\frac{d^2x}{dt^2} = -P'\frac{x}{\rho} - T'\frac{y}{\rho},$$
$$M\frac{d^2y}{dt^2} = -P'\frac{y}{\rho} + T'\frac{x}{\rho},$$
$$M\frac{d^2z}{dt^2} = -S';$$

or, putting P, T, S for $\frac{P'}{M}$, $\frac{T'}{M}$, $\frac{S'}{M}$,

$$\frac{d^2x}{dt^2} = -P\frac{x}{\rho} - T\frac{y}{\rho}, \quad \ldots\ldots\ldots(1)$$
$$\frac{d^2y}{dt^2} = -P\frac{y}{\rho} + T\frac{x}{\rho}, \quad \ldots\ldots\ldots(2)$$
$$\frac{d^2z}{dt^2} = -S \quad \ldots\ldots\ldots\ldots\ldots(3).$$

From (1) and (2)

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = T\rho.$$

$$\text{Let } x\frac{dy}{dt} - y\frac{dx}{dt} = h;$$

$$\text{then } \frac{dh}{dt} = T\rho,$$

$$\text{and } h = \rho^2 \frac{d\theta}{dt};$$

$$\therefore h\frac{dh}{d\theta} = T\rho^3 = \frac{T}{u^3},$$

putting u for $\frac{1}{\rho}$;

$$\therefore h^2 = h_1^2 + 2\int \frac{T d\theta}{u^3}, \ldots\ldots\ldots (4)$$

$$\text{also } x = \frac{\cos\theta}{u};$$

$$\therefore \frac{dx}{dt} = -\left(\frac{\cos\theta}{u^2}\frac{du}{d\theta} + \frac{\sin\theta}{u}\right)\frac{d\theta}{dt}$$

$$= -h(\cos\theta\frac{du}{d\theta} + u\sin\theta),$$

$$\frac{d^2x}{dt^2} = -\frac{dh}{dt}(\cos\theta\frac{du}{d\theta} + u\sin\theta) - h(\cos\theta\frac{d^2u}{d\theta^2} + u\cos\theta)\frac{d\theta}{dt}$$

$$= -\cos\theta\frac{T}{u}\frac{du}{d\theta} - T\sin\theta - h^2u^2\cos\theta\left(\frac{d^2u}{d\theta^2} + u\right).$$

But $\frac{d^2x}{dt^2} = -P\cos\theta - T\sin\theta$;

$$\therefore \frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} + \frac{T}{h^2u^3}\frac{du}{d\theta} = 0,$$

or, substituting for h^2 from (4),

$$\frac{d^2u}{d\theta^2} + u - \frac{\frac{P}{h_1^2u^2} - \frac{T}{h_1^2u^3}\frac{du}{d\theta}}{1 + 2\int\frac{T d\theta}{h_1^2u^3}} = 0 \ldots\ldots\ldots (5)$$

$$\text{and } \frac{dt}{d\theta} = \frac{1}{hu^2} = \frac{1}{h_1u^2\sqrt{1 + 2\int\frac{T d\theta}{h_1^2u^3}}} \ldots\ldots\ldots\ldots (6)$$

Again, let $\tan PON = s$.

$$\text{Then } z = \rho s = \frac{s}{u},$$

$$\frac{d^2z}{dt^2} = -S.$$

Combining these two with equations (5) and (6), we get

$$\frac{d^2s}{d\theta^2} + s + \frac{\frac{S - Ps}{h_1^2u^2} + \frac{T}{h_1^2u^3}\frac{du}{d\theta}}{1 + 2\int\frac{T d\theta}{h_1^2u^3}} = 0 \ldots\ldots (7)$$

These equations though perfectly general are of little use except when P, T and S are of particular forms. Their principal application is in determining the motion of the moon about the earth when acted on by the disturbing force of the sun. In this case they are integrated by a series of approximations.

64. When the forces X, Y, Z depend on the velocity of the particle as well as its position, the integration of the equations of motion can seldom be completely effected. It frequently happens, however, that they can be put into a form which will enable us, by means of laborious calculation, to determine many points connected with the motion to any required degree of approximation.

Let us take for an example the motion of a body in the air acted on by gravity. The resistance of the air varies as the square of the velocity of the body, so that the equations of motion are

$$\frac{d^2x}{dt^2} = -kv^2\frac{dx}{ds},$$

$$\frac{d^2y}{dt^2} = -g - kv^2\frac{dy}{ds}.$$

These may be written

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt}\frac{ds}{dt} = 0, \quad \ldots\ldots\ldots(1)$$

$$\frac{d^2y}{dt^2} + k\frac{dy}{dt}\frac{ds}{dt} + g = 0. \quad \ldots\ldots\ldots(2)$$

The integral of equation (1) is

$$\frac{dx}{dt} = Ce^{-ks}.$$

Let V be the initial velocity of the body, α the angle its direction makes with the horizon, then

$$\frac{dx}{dt} = V\cos\alpha e^{-ks} \ldots\ldots\ldots(3)$$

Let p be written for $\frac{dy}{dx}$, then $\frac{dy}{dt} = p\frac{dx}{dt}$; substituting this in equation (2), and reducing by means of (1), we have

$$\frac{dx}{dt}\frac{dp}{dt} = -g.$$

$$\therefore \frac{dp}{dx} = -\frac{g}{V^2\cos^2\alpha}e^{2ks}. \quad \ldots\ldots\ldots\ldots (4)$$

$$\text{Now, } \sqrt{1+p^2} = \frac{ds}{dx};$$

$$\therefore \sqrt{1+p^2}\frac{dp}{dx} = -\frac{g}{V^2\cos^2\alpha}e^{2ks}\frac{ds}{dx}.$$

The integral of this equation is

$$p\sqrt{1+p^2}+\log\left(p+\sqrt{1+p^2}\right)=\gamma-\frac{g}{kV^2\cos^2\alpha}\epsilon^{2kx}, \quad \ldots(5)$$

where γ is a constant introduced by integration, and is given by the equation

$$\tan\alpha\sec\alpha+\log\left(\tan\alpha+\sec\alpha\right)=\gamma-\frac{g}{kV^2\cos^2\alpha}.$$

From equations (4) and (5)

$$k\frac{dx}{dp}=\frac{-1}{\gamma-p\sqrt{1+p^2}-\log\left(p+\sqrt{1+p^2}\right)}; \quad \ldots\ldots\ldots(6)$$

$$\therefore\ k\frac{dy}{dp}=\frac{-p}{\gamma-p\sqrt{1+p^2}-\log\left(p+\sqrt{1+p^2}\right)}; \quad \ldots\ldots\ldots(7)$$

$$\text{and } \sqrt{kg}\,\frac{dt}{dp}=\frac{-1}{\left\{\gamma-p\sqrt{1+p^2}-\log\left(p+\sqrt{1+p^2}\right)\right\}^{\frac{1}{2}}}; \quad \ldots\ldots(8)$$

these equations cannot be integrated in finite terms; they give, however, the values of x, y and t corresponding to any value of p, in the form of a definite integral, whose value may be found to any degree of approximation by the ordinary methods applicable in such cases. Thus any number of points in the path of the body, and the respective times of reaching those points may be determined. The velocity corresponding to any value of p is given by the equation

$$v^2=\frac{g}{k}\cdot\frac{1+p^2}{\gamma-p\sqrt{1+p^2}-\log\left(p+\sqrt{1+p^2}\right)}. \quad \ldots\ldots\ldots(9)$$

When $p=0$ the body is at the highest point of its course, and at that time

$$v^2=\frac{g}{\gamma k}.$$

Equation (4) may be written

$$\frac{d^2y}{dx^2}=-\frac{g}{V^2\cos^2\alpha}\epsilon^{2kx},$$

which is the differential equation to the path of the body. We may shew that this curve has a vertical asymptote. Since p becomes negative and continues to increase in actual magnitude, let p_1 be a very great value of p, and x_1 the corresponding value of x; then in equation (6), neglecting all powers of p except the highest, we have approximately for values of p greater than p_1

$$k\frac{dx}{dp}=\frac{1}{p^2};$$

$$\therefore\ kx = kx_1 + \frac{1}{p_1} - \frac{1}{p};$$

which shews that x is finite when $p = -\infty$.

Hence then $-\frac{1}{k}\int_{\tan a}^{-\infty} \frac{dp}{\gamma - p\sqrt{1+p^2} - \log(p+\sqrt{1+p^2})}$ is finite;

let this equal a, then x is always less than a, and approaches a as a limit when p increases indefinitely; a line parallel to the axis of y at a distance a from the origin is therefore an asymptote to the curve.

65. Let us take as another example the case of a body describing a nearly circular orbit round a center of force varying inversely as the square of the distance, in a resisting medium of very small density.

The polar equations of motion are

$$\frac{d^2u}{d\theta^2} + u - \frac{\frac{P}{h_1^2u^2} - \frac{t}{h_1^2u^3}\frac{du}{d\theta}}{1 + 2\int \frac{T d\theta}{h_1^2 u^3}} = 0, \quad \ldots\ldots\ldots (1)$$

$$\frac{d\theta}{dt} = h_1 u^2 \sqrt{1 + 2\int \frac{T d\theta}{h_1^2 u^3}}. \quad \ldots\ldots\ldots\ldots (2)$$

In this case if kv^2 is the resistance, k is very small, and

$$P = \mu u^2 - \frac{k}{u^4}\frac{du}{d\theta}\sqrt{u^2 + \left(\frac{du}{d\theta}\right)^2}\left(\frac{d\theta}{dt}\right)^2, \quad \ldots\ldots (3)$$

$$T = -\frac{k}{u^3}\sqrt{u^2 + \left(\frac{du}{d\theta}\right)^2}\left(\frac{d\theta}{dt}\right)^2. \quad \ldots\ldots\ldots\ldots (4)$$

Let $\frac{\mu}{h_1^2} = a$, $\frac{\mu^2}{h_1^3} = n$, then

$$\frac{T}{h_1^2 u^3} = -\frac{k a^4}{h_1^3 u^6}\sqrt{u^2 + \left(\frac{du}{d\theta}\right)^2}\left(\frac{d\theta}{dt}\right)^2, \quad \ldots\ldots (5)$$

$$\frac{P}{h_1^2 u^3} - \frac{T}{h_1^2 u^3}\frac{du}{d\theta} = a,$$

and equation (1) becomes

$$\frac{d^2u}{d\theta^2} + u - \frac{a}{1 + 2\int \frac{T d\theta}{h_1^2 u^3}} = 0. \quad \ldots\ldots\ldots (6)$$

Neglecting small quantities of the first order, we have

$$u = a, \quad \frac{d\theta}{dt} = n.$$

Substituting these values and neglecting small quantities of the second order,

$$\frac{T}{h_1^2 u^2} = -\frac{k}{a};$$

$$\therefore \int \frac{T d\theta}{h_1^2 u^2} = -\frac{k\theta}{a},$$

$$\frac{d^2u}{d\theta^2} + u - a - 2k\theta = 0;$$

$$\therefore u = a\{1 + \frac{2k\theta}{a} + e\cos(\theta - \alpha)\},$$

where e is a small quantity, the orbit being supposed nearly circular;

$$\frac{d\theta}{dt} = n\{1 + \frac{3k\theta}{a} + 2e\cos(\theta - \alpha)\}. \ldots\ldots (7)$$

Substituting these values and neglecting small quantities of the third order,

$$\frac{T}{h_1^2 u^2} = -\frac{k}{a}\{1 - \frac{4k\theta}{a} - e\cos(\theta - \alpha)\};$$

$$\therefore 2\int \frac{T d\theta}{h_1^2 u^2} = -\frac{2k}{a}\{\theta - \frac{2k\theta^2}{a} - e\sin(\theta - \alpha)\},$$

$$\frac{d^2u}{d\theta^2} + u - a - 2k\theta + 2ke\sin(\theta - \alpha) = 0;$$

$$\therefore u = a\{1 + \frac{2k\theta}{a} + e\cos(\theta - \alpha) + \frac{k\theta}{a}e\cos(\theta - \alpha)\},$$

$$\frac{d\theta}{dt} = n\{1 + \frac{3k\theta}{a} + \frac{3k^2\theta^2}{2a^2} + 2e\cos(\theta - \alpha) + \frac{4k\theta}{a}e\cos(\theta - \alpha)$$

$$+ \frac{k}{a}e\sin(\theta - \alpha) + e^2\cos^2(\theta - \alpha)\}.$$

By repeating the process the solution might be carried to a higher degree of approximation; the labour, however, increases very rapidly. This example has been selected to exhibit the use of the equations of Art. 53; and also as an example of the method of solution by successive approximation. Since terms multiplied by θ occur in the expression for u, this expression will cease to be an approximation at all when θ becomes large.

To express θ in terms of t, we must proceed in a similar manner:

$$\frac{d\theta}{dt} = n; \quad \therefore \theta = nt + \epsilon.$$

Substituting this value of θ in equation (7)

$$\frac{\delta\theta}{dt} = n\{1 + \frac{3k}{a}.(nt + \epsilon) + 2e\cos(nt + \epsilon - \alpha)\};$$

$$\therefore\ \theta = nt + \epsilon + \frac{3k}{2a}(nt^2 + 2\epsilon t) + 2e\sin(nt + \epsilon - \alpha)$$

correctly to the first order of small quantities: by substituting this value of θ in the next value of $\frac{d\theta}{dt}$ and integrating, we might obtain a more approximate value of θ.

66. In the problems which have been hitherto considered the forces which acted on the particle were known in terms of the position, velocity, and direction of motion of the particle, which was perfectly free to move as these forces would cause it to move; in other words, in the cases in which we have integrated the equations

$$m\frac{d^2x}{dt^2} = X, \quad m\frac{d^2y}{dt^2} = Y, \quad m\frac{d^2z}{dt^2} = Z,$$

X, Y, and Z have been given functions of x, y, z and their first differential coefficients. And x, y, and z have been subject to no other relations than those furnished by the above equations. In this case we have three equations which are theoretically sufficient to determine the three unknown quantities x, y, z in terms of t; or by eliminating t to give two equations between x, y, and z.

There is, however, an extensive class of problems in which X, Y, and Z contain, in addition to given forces, unknown forces arising from geometrical conditions to which the motion is subject, such, for instance, as the unknown tension of a string, the unknown reaction of some curve, or surface. These forces will enter as unknown quantities in the three equations of motion, but however many unknown quantities of this sort may appear, there will always be just so many additional equations expressing these geometrical relations.

The methods to be pursued in cases of this sort will of course depend on the particular geometrical conditions imposed on the particle, they will, however, be generally very similar to those pursued in other cases. The principal problems of this class are those in which a particle is constrained to move on a particular surface, or some particular curve line. We will first prove some properties, and deduce some equations which are common to all problems of this sort, and, at the same time, point out the methods to be pursued in particular cases.

Let us first consider the case where the motion is in one plane.

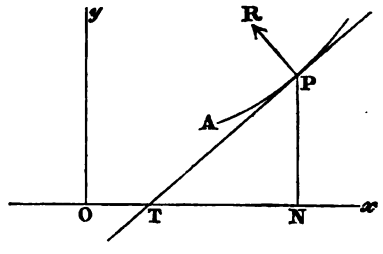
67. Let a particle be constrained to move on a smooth plane curve and be acted on by any given forces in the plane of the curve.

By the term smooth we understand that the curve is incapable of exercising any reaction except in the direction of the normal.

Let $y = f(x)$(1)

be the equation to the curve; X, Y the resolved parts parallel to the axes of the forces which act on the particle at the time t. First let these forces be functions of x, y and constant quantities only; R the reaction of the curve, A any fixed point in the curve, P the position of the particle at the time t,

$AP = s$, $PTN = \theta$.



The equations of motion are

$$m\frac{d^2x}{dt^2} = X - R\sin\theta,$$

$$m\frac{d^2y}{dt^2} = Y + R\cos\theta,$$

or since $\sin\theta = \frac{dy}{ds}$, $\cos\theta = \frac{dx}{ds}$.

These equations may be written

$$m\frac{d^2x}{dt^2} = X - R\frac{dy}{ds} \text{.....(2)}$$

$$m\frac{d^2y}{dt^2} = Y + R\frac{dx}{ds} \text{.....(3)}$$

$$y = f(x) \text{.....(1)}$$

Eliminating R from (2) and (3) we have

$$m\left\{\frac{dx}{ds}\frac{d^2x}{dt^2} + \frac{dy}{ds}\frac{d^2y}{dt^2}\right\} = X\frac{dx}{ds} + Y\frac{dy}{ds},$$

or $$m\left\{\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2}\right\} = X\frac{dx}{dt} + Y\frac{dy}{dt} \text{ ... (4)}$$

When X and Y are such functions of x and y that $Xdx + Ydy$ is the differential of some function $F(x, y)$ of x and y considered as independent, that is, when X and Y satisfy the condition

$$\frac{dX}{dy} = \frac{dY}{dx},$$

we may integrate (4) without reference to (1).

$$\text{Since } v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

That equation may be written

$$mv\frac{dv}{dt} = m\frac{ds}{dt}\frac{d^2s}{dt^2} = \frac{d.F(x,y)}{dt}.$$

The integral of which is

$$mv^2 = 2F(x,y) + C.$$

Let v_1, x_1, y_1, be corresponding values of v, x, y, then

$$mv_1^2 = 2F(x_1, y_1) + C,$$

$$\text{and } \therefore\ m(v^2 - v_1^2) = 2F(x,y) - 2F(x_1, y_1) \quad \ldots\ldots\ (5)$$

This equation shews that the change of velocity in passing from the point (x_1y_1) to the point (xy) is independent of the form of the curve along which the particle has moved, and by a reference to Art. 50 it will be seen that this change is the same as if the body had been free and had moved from the one point to the other under the action of the forces X and Y.

When X and Y do not satisfy the above condition, this is not the case. In that case we may integrate (4) by eliminating x or y from the right-hand side by means of equation (1). These cases, however, seldom occur, and are of little importance.

Equation (4) may be put under the form,

$$m\frac{d^2s}{dt^2} = X\frac{dx}{ds} + Y\frac{dy}{ds},$$

$$\text{or } m\frac{d^2s}{dt^2} = X\cos\theta + Y\sin\theta.$$

The right-hand side of this equation is the resolved part of the forces X, Y in the direction of the tangent: calling this T the equation becomes

$$m\frac{d^2s}{dt^2} = T \quad \ldots\ldots\ldots\ (6)$$

This form of the equation is useful in cases where T can be expressed easily as a function of s.

Again, from equation (2) and (3) we have

$$m\left\{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right\} = Y\frac{dx}{dt} - X\frac{dy}{dt} + R\frac{ds}{dt},$$

$$\text{or, } R = X\sin\theta - Y\cos\theta + m\frac{dt}{ds}\left\{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right\}.$$

Now if ρ is the radius of curvature at the point P

$$\rho = \frac{\left(\frac{ds}{dt}\right)^3}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}};$$

$$\therefore R = X\sin\theta - Y\cos\theta + \frac{m}{\rho}\left(\frac{ds}{dt}\right)^2.$$

The first terms on the right-hand side of this equation are the resolved parts of the forces X and Y along the normal: if we call the sum of these N, the equation becomes

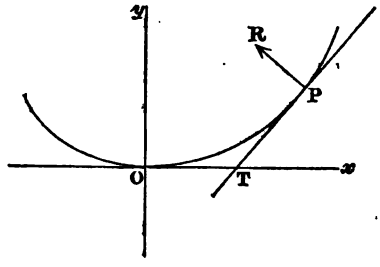
$$R + N = \frac{mv^2}{\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

In this equation ρ is the absolute length of the radius of curvature, and is always positive; R is the pressure exerted by the curve on the particle in a direction towards the centre of curvature, and N is estimated in the same direction.

We have supposed that the curve is capable of exerting a pressure on the particle in direction of the normal, either outwards or inwards, as is the case when a particle is enclosed in a small tube; if, however, the particle moves on the curve, R is restricted to being only positive or only negative, according as the particle moves on the concave or convex side of the curve; and in either of these cases when R becomes zero and changes sign, at that point the particle leaves the curve and the motion becomes that of a free particle.

68. As an example, let us take the case of a body moving on a smooth curve under the action of gravity.

Let the lowest point O of the curve be taken as origin, and the axis of x be horizontal, and let $OP = s$. The equations of motion will be



$$m\frac{d^2x}{dt^2} = -R\frac{dy}{ds}, \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$m\frac{d^2y}{dt^2} = -mg + R\frac{dx}{ds}, \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$y = f(x) \quad \ldots\ldots\ldots\ldots\ldots (3)$$

Proceeding here as in the general case, we have

$$2m\frac{ds}{dt}\frac{d^2s}{dt^2}=-2mg\frac{dy}{dt};$$

$$\therefore\ mv^2=C-2mgy.$$

Let v_1 be the value of v when $y=y_1$, then we have

$$m(v^2-v_1^2)=2mg(y_1-y),$$

$$\text{or } v^2-v_1^2=2g(y_1-y).$$

This equation shews that the change of velocity will be the same as if the body had moved through the same vertical height freely. It will be useful to bear this result in mind.

It is sometimes convenient to put the preceding equation under the form

$$\frac{d^2s}{dt^2}=-g\frac{dy}{ds}.$$

This is useful when $\frac{dy}{ds}$ can be expressed easily as a function of s. When this is the case this equation gives at once the motion along the curve in terms of the time.

To find the pressure exercised by the curve, we have

$$R=mg\frac{dx}{ds}+\frac{v^2}{\rho}.$$

Suppose the body to fall from rest at a point where $y=h$, the equation

$$v^2-v_1^2=2g(y_1-y)$$

$$\text{becomes } v^2=2g(h-y);$$

and if u be the velocity at the lowest point

$$u^2=2gh,$$

the body will move on and ascend the other side of the curve, the velocity at any point being still given by the equation

$$v^2=2g(h-y),$$

which will become zero when $y=h$. Or, whatever the form of the curve, the body will rise to the same height as that from which it started.

It is easily seen that the body will oscillate, the points of rest being always at the same height above the horizontal line.

69. Let the curve be an inverted cycloid, whose equation is

$$x = a \operatorname{vers}^{-1} \frac{y}{a} + \sqrt{2ay - y^2}.$$

From which we have $\frac{dx}{dy} = \sqrt{\frac{2a-y}{y}}$;

$$\therefore \frac{ds}{dy} = \sqrt{\frac{2a}{y}};$$

from which we have $s = 2\sqrt{2ay}$;

$$\text{and } \frac{dy}{ds} = \frac{s}{4a}.$$

Substituting this in the equation

$$\frac{d^2s}{dt^2} = -g\frac{dy}{ds},$$

$$\text{we have } \frac{d^2s}{dt^2} + \frac{gs}{4a} = 0.$$

The integral of this equation is

$$s = A \cos\left(\sqrt{\frac{g}{4a}}\, t + B\right);$$

$$\text{and } \therefore \frac{ds}{dt} = -A\sqrt{\frac{g}{4a}} \cdot \sin\left(\sqrt{\frac{g}{4a}}\, t + B\right).$$

Suppose the body to fall from rest from a point where $s = s_1$, we have then

$$s_1 = A \cos B,$$

$$0 = -A\sqrt{\frac{g}{4a}} \sin B;$$

$$\therefore B = 0, \qquad A = s_1,$$

$$\text{and } s = s_1 \cos \sqrt{\frac{g}{4a}}\, t,$$

when $s = 0$, we must have

$$\sqrt{\frac{g}{4a}}\, t = (2n+1)\frac{\pi}{2}.$$

Hence the time of reaching the lowest point is $\pi\sqrt{\frac{a}{g}}$;

and the whole time from rest to rest again is $2\pi\sqrt{\frac{a}{g}}$. These are both independent of the length s_1.

Hence we see that from whatever point the body falls it will always reach the lowest point in the same time. On this account the cycloid is called the "Tautochronous curve."

The equation $\frac{ds}{dt} = -s_1 \sqrt{\frac{g}{4a}} \sin \sqrt{\frac{g}{4a}}\, t$

gives the velocity at any time; and since we know the position at any time, we can find the velocity in any position. It may however be more readily found from the equation

$$v^2 = 2g\,(h-y),$$

for we have $s_1^2 = 8\,a\,h,$

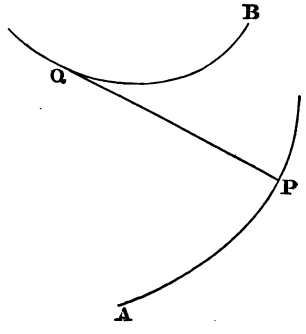
$$s^2 = 8\,a\,y,$$

$$\therefore\ v^2 = \frac{g}{4a}\,(s_1^2 - s^2).$$

The pressure on the curve is given by the equation

$$R = mg\frac{dx}{ds} + \frac{mv^2}{\rho}.$$

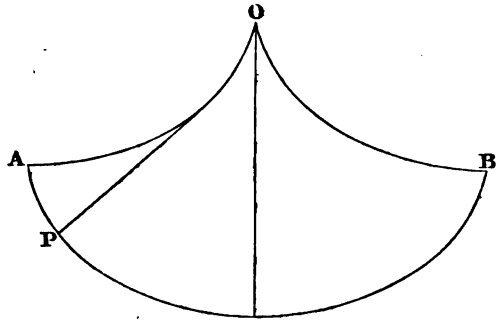
Practically it is impossible to obtain a perfectly smooth curve; the properties of the evolute, however, enable us to produce the same dynamical effect.



Suppose AP to be the curve in which it is required that the particle should oscillate, and that QB is its evolute. If a string be unwrapped from BQ, its extremity will trace out the curve PA, and since the only dynamical effect of the curve PA is to exercise a resistance in the direction PQ of its normal, this effect will be equally produced by the string PQ exercising a tension; and therefore the dynamical circumstances of the body will be the same when moving on the curve, and when attached to the string.

Now, in the case of the cycloid, the evolute is two semi-cycloids in the position in the figure.

Hence, if a string OP of length $4a$ is fastened at O, and the body P swings backwards and forwards so that the string wraps and



unwraps itself on the curves OA, OB, the body P will describe the curve AB in the manner we have just investigated. Such an instrument is called a cycloidal pendulum.

70. Let us now consider the motion of a body in a circular arc. The equation to the circle is

$$x^2 = 2ay - y^2,$$

from which we have $\dfrac{dx}{dy} = \dfrac{a-y}{x}$;

$$\therefore \frac{ds}{dy} = \frac{a}{x} = \frac{a}{\sqrt{2ay-y^2}}.$$

$$s = a \operatorname{vers}^{-1}\frac{y}{a};$$

$$\therefore y = a \operatorname{vers}\frac{s}{a}.$$

$$\frac{dy}{ds} = \sin\frac{s}{a}.$$

Hence the equation becomes

$$\frac{d^2s}{dt^2} = -g\sin\frac{s}{a};$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = 2ag\cos\frac{s}{a} + C.$$

If s_1 be the value of s when the body is at rest, we have

$$0 = 2ag\cos\frac{s_1}{a} + C;$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = 2ag\left(\cos\frac{s}{a} - \cos\frac{s_1}{a}\right).$$

We might also have obtained the velocity from the equation

$$2m\frac{ds}{dt}\frac{d^2s}{dt^2} = -2mg\frac{dy}{dt},$$

$$\text{from which } \left(\frac{ds}{dt}\right)^2 = 2g(h-y).$$

Neither of these equations can be integrated again in finite terms. The time of reaching the lowest point may, however, be found from them in a series which converges very rapidly when h is small.

If we require the time to a first approximation only, we can find it at once from the equation

$$\frac{d^2s}{dt^2} = -g\sin\frac{s}{a},$$

which is to a first approximation

$$\frac{d^2s}{dt^2} + \frac{g}{a}s = 0;$$

$$\therefore\ s = s_1\cos\sqrt{\frac{g}{a}}.t,$$

and the time of a small oscillation will be $\pi\sqrt{\frac{a}{g}}$, the same as in a cycloid where the length of string is a.

When the oscillations are not small though we cannot find the position at any time, we know the velocity in any position from the equation

$$v^2 = 2g(h-y),$$

if u be the velocity at the lowest point

$$u^2 = 2gh,$$

$$\text{and } v^2 = u^2 - 2gy.$$

If θ be the angular distance from the lowest point,

$$y = a - a\cos\theta;$$

$$\therefore\ v^2 = u^2 - 2ag + 2ag\cos\theta.$$

$$\text{Also, } R = m\frac{v^2}{a} + mg\frac{dx}{ds}$$

$$= m\frac{v^2}{a} + mg\cos\theta$$

$$= m\frac{u^2}{a} - 2mg + 3mg\cos\theta.$$

Hence at the lowest point

$$R = m\frac{u^2}{a} + mg,$$

at highest point

$$R = m\frac{u^2}{a} - 5mg,$$

when the string is horizontal

$$R = m\frac{u^2}{a} - 2mg.$$

If R becomes zero and changes its sign, the body will leave the curve. Now if u^2 is greater than $5ag$ we shall never have $R=0$, since the least value of R is

$$m\left(\frac{u^2}{a} - 5g\right).$$

Also, the least value of v^2 is $u^2 - 4ag$, so that the particle will never come to rest, but will continue moving round and round.

If $u^2 = 5ag$ we shall have $R = 0$ at the highest point, but it will not change sign and therefore the body will not leave the curve. Also v will never become zero, and therefore the body will move round and round. If however u^2 is less than $5ag$, R will become zero before the body reaches the highest point, and the corresponding value of θ will be given by the equation

$$\cos\theta = -\frac{u^2 - 2ag}{3ag}.$$

When u^2 is greater than $2ag$ this value of $\cos\theta$ is negative; and as θ increases, $\cos\theta$ will increase negatively, so that R would change sign. The particle will leave the curve therefore at this point, and its motion will be that of a body projected freely.

That it will reach this point is shewn by the equation

$$v^2 = u^2 - 2ag + 2ag\cos\theta,$$

which gives a greater value of θ for the point at which v would vanish, namely, $\cos\theta = -\dfrac{u^2 - 2ag}{2ag}$.

If $u^2 = 2ag$ the values of θ, for which $R = 0$ and $v = 0$ are coincident, θ therefore will not increase, and R will not become negative. Hence in this case the body will not leave the curve. If u^2 is less than $2ag$, we have $v = 0$ for a smaller value of θ than that which would make $R = 0$, R therefore will never vanish.

We have dwelt on this case at considerable length, not from any importance in the question itself; but because it exhibits very clearly

the mode of interpreting the equations to be adopted in all cases of this sort.

We have two quantities R and v^2, each a function of the angular distance θ from the bottom, and each from its nature incapable of becoming negative; the angle θ will go on increasing till one or both become zero; and some change will take place in the motion when either of these happens. If $v^2 = 0$ before $R = 0$, the motion will be continuous, but θ will have attained its maximum and will begin to diminish. If $R = 0$ first and an increase of θ would make R negative there will be a discontinuity in the motion.

If no value of θ makes either $R = 0$ or $v = 0$, the angle θ will go on increasing.

These remarks as far as the detail is concerned apply only to this case, but are sufficiently general to exhibit the manner of interpreting the equations in other cases.

71. We will now consider the case where the motion is on a curve of double curvature. The reaction of the curve will in this case be in the normal plane, but we cannot assign, *a priori*, in what direction in that plane.

Let R be the reaction, l, m, n the direction-cosines of the line in which it acts.

Then the equations of motion are

$$\left.\begin{aligned} M\frac{d^2x}{dt^2} &= X + Rl, \\ M\frac{d^2y}{dt^2} &= Y + Rm, \\ M\frac{d^2z}{dt^2} &= Z + Rn. \end{aligned}\right\} \ldots\ldots\ldots (1)$$

l, m, n are connected by the equation

$$l^2 + m^2 + n^2 = 1 \ldots\ldots\ldots (2)$$

and also by the equation expressing that the direction of R is perpendicular to the tangent. This equation is

$$l\frac{dx}{ds} + m\frac{dy}{ds} + n\frac{dz}{ds} = 0 \ldots\ldots\ldots (3)$$

These five equations together with the two equations to the curve are sufficient to determine x, y, z, l, m, n, and R at any instant.

Eliminating R from equations (1) by means of (3) we have

$$2M\left\{\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}\right\} = 2\left\{X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right\};$$

$$\text{or } M\frac{d.v^2}{dt} = 2\left\{X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right\} \ldots\ldots\ldots (4).$$

If $Xdx + Ydy + Zdz$ is a perfect differential of some function of x, y, z considered as independent variables, that is, if X, Y, Z satisfy the three equations

$$\frac{dX}{dy} = \frac{dY}{dx}, \quad \frac{dX}{dz} = \frac{dZ}{dx}, \quad \frac{dY}{dz} = \frac{dZ}{dy},$$

equation (4) can be integrated without making use of the relation between x, y and z given by the equations to the curve.

Let this function be $M.F(x, y, z)$ then we shall have

$$v^2 = C + 2F(x, y, z),$$

$$\text{and } v^2 - v_1^2 = 2F(x, y, z) - 2F(x_1, y_1, z_1),$$

or the change of velocity in passing from the point $(x_1 y_1 z_1)$ to the point (xyz) will depend only on the positions of those points, and not on the nature of the curve between them: and this change of velocity will be the same as would have taken place if the particle had moved freely from one point to the other.

We also see that the velocity is the same at all points at which the value of $F(x, y, z)$ is the same, or

$$F(x, y, z) = C$$

is the equation to a surface through which the particle will pass with the same velocity at whatever point of the surface it arrives, and by whatever path it has travelled, provided only that it started from the surface $F(x, y, z) = C_1$ with the same velocity v_1.

Or the change of velocity in passing from any surface

$$F(x, y, z) = C_1$$

to any surface $$F(x, y, z) = C_2$$

is the same, whatever the path between those surfaces.

If $Xdx + Ydy + Zdz$ is not a perfect differential of three independent variables, we cannot perform the integration without introducing the relations between x, y and z given by the equations to the curve; these equations will therefore modify the form of expression for the velocity, and the preceding results will not follow.

Equation (4) may also be put into the form

$$M\frac{d^2s}{dt^2} = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = T,$$

where T is the sum of the resolved parts in direction of the tangent of the forces X, Y, Z.

In cases where this resolved part can be expressed easily in terms of s this will give the motion along the curve at once. In the case

where no forces act we see that the velocity is constant. In this case we can easily shew that the whole reaction is in the osculating plane of the curve. The equations of motion in that case are

$$M\frac{d^2x}{dt^2} = lR,$$

$$M\frac{d^2y}{dt^2} = mR,$$

$$M\frac{d^2z}{dt^2} = nR,$$

$$l\frac{dx}{ds} + m\frac{dy}{ds} + n\frac{dz}{ds} = 0.$$

Now the direction-cosines of the normal to the osculating plane are proportional to

$$\frac{d^2y}{dt^2}\frac{dz}{dt} - \frac{d^2z}{dt^2}\frac{dy}{dt}, \quad \frac{d^2z}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dz}{dt}, \quad \frac{d^2x}{dt^2}\frac{dy}{dt} - \frac{d^2y}{dt^2}\frac{dx}{dt};$$

and therefore, considering the equations of motion, proportional to

$$\left(m\frac{dz}{dt} - n\frac{dy}{dt}\right), \quad \left(n\frac{dx}{dt} - l\frac{dz}{dt}\right), \quad \left(l\frac{dy}{dt} - m\frac{dx}{dt}\right),$$

and if θ be the angle between this line and the direction of R's action, the numerator of $\cos\theta$ is

$$l\left(m\frac{dz}{dt} - n\frac{dy}{dt}\right) + m\left(n\frac{dx}{dt} - l\frac{dz}{dt}\right) + n\left(l\frac{dy}{dt} - m\frac{dx}{dt}\right),$$

which $= 0$,

and $\therefore\ \theta = \frac{\pi}{2}$,

or R acts in the osculating plane.

72. Let us next consider the motion of a particle which is constrained to move on a given smooth surface.

In this case the whole reaction of the surface will be in the direction of the normal.

Let $u = 0$, ...(1)

be the equation to the surface, and let l, m, n be the direction-cosines of the normal at any point, then the equations of motion are

$$M\frac{d^2x}{dt^2} = X + Rl, \quad \ldots\ldots\ldots (2)$$

$$M\frac{d^2y}{dt^2} = Y + Rm, \quad \ldots\ldots\ldots (3)$$

$$M\frac{d^2z}{dt^2} = Z + Rn \ldots\ldots\ldots (4)$$

l, m, and n are known functions of x, y, and z, determined by equation (1), and these four equations will be sufficient to determine x, y, z and R as functions of t.

We shall have, as before,

$$2Mv\frac{dv}{dt} = 2\left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right),$$

and the same results may be deduced from this equation as were deduced in the case of a curve line.

To find the equation which must be joined to (1) to determine the path of the particle on the surface, we have

$$l = \frac{\dfrac{du}{dx}}{\sqrt{\left(\dfrac{du}{dx}\right)^2 + \left(\dfrac{du}{dy}\right)^2 + \left(\dfrac{du}{dz}\right)^2}},$$

$$m = \frac{\dfrac{du}{dy}}{\sqrt{\left(\dfrac{du}{dx}\right)^2 + \left(\dfrac{du}{dy}\right)^2 + \left(\dfrac{du}{dz}\right)^2}},$$

$$n = \frac{\dfrac{du}{dz}}{\sqrt{\left(\dfrac{du}{dx}\right)^2 + \left(\dfrac{du}{dy}\right)^2 + \left(\dfrac{du}{dz}\right)^2}}.$$

Eliminating R from the equations (2), (3), (4), we shall have two equations between x, y, z, and t; and these combined with (1) will give x, y and z as functions of t: or will give another relation between x, y and z by eliminating t; R can then be found from any one of the previous equations.

We can shew that whenever there are no forces acting on the body, the osculating plane of its path will always pass through the normal to the surface. The equations are

$$M\frac{d^2x}{dt^2} = Rl,$$

$$M\frac{d^2y}{dt^2} = Rm,$$

$$M\frac{d^2z}{dt^2} = Rn.$$

And, as before, we can shew that the direction-cosines of the normal to the osculating plane are proportional to

$$\left(m\frac{dz}{dt}-n\frac{dy}{dt}\right),\ \left(n\frac{dx}{dt}-l\frac{dz}{dt}\right),\ \left(l\frac{dy}{dt}-m\frac{dx}{dt}\right),$$

and it is therefore at right angles to the normal.

73. In the particular case where the proposed surface is one of revolution, some of the preceding equations may be put under a simpler form.

Let the axis of revolution be the axis of z, and let $r=f(z)$ be the equation to the generating curve.

$$x=r\cos\theta,\ \ y=r\sin\theta.$$

The direction of the reaction R will in this case always pass through the axis.

From the first two equations of motion we have

$$M\left\{x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right\}=xY-yX.$$

When $xY-yX=0$ the integral of this equation is

$$r^2\frac{d\theta}{dt}=h.$$

This will be the case when $X=0$, and $Y=0$, or when the whole force is parallel to the axis of revolution; it will also be the case when the resultant of X, Y and Z passes through the axis of revolution.

Let ϕ be the angle which the direction of motion makes with the generating curve through the particle. Then we shall have

$$\sin\phi=\frac{r}{v}\frac{d\theta}{dt}=\frac{h}{vr}.$$

If no forces act, v will be constant, and $\sin\phi$ will vary inversely as the distance from the axis.

Again, suppose the axis of revolution to be vertical and the only force acting to be gravity: we then have

$$v^2=2g(k-z),$$

$$\text{which gives } \sin\phi=\frac{h}{r\sqrt{2g(k-z)}}.$$

In the cases of constraint that we have considered the curves and surfaces on which we supposed the particle compelled to move were fixed in space. We might vary the conditions by supposing them in motion, either according to some given law, or in some manner due to the action of the particle. Some of these cases

will occupy our attention afterwards. We have here given sufficient to shew how such problems are to be treated when they occur.

74. We will conclude this chapter by proving in the case of the motion of a particle, a principle of very general application, called "The Principle of Least Action."

When a free particle moves under the action of forces X, Y, Z, the equations of motion are

$$M\frac{d^2x}{dt^2}=X,\quad M\frac{d^2y}{dt^2}=Y,\quad M\frac{d^2z}{dt^2}=Z.....(1)$$

$$\text{and if}\quad 2Xdx+2Ydy+2Zdz=Mdf(x,y,z)...(2)$$

$$\text{we have}\quad v^2=V^2+f(x,y,z)-f(a,b,c)\ ...(3)$$

Let $\int v ds$ be taken between limits corresponding to two fixed points in the path of the body; then this integral is less than the corresponding integral would be if the particle were compelled to describe any other path by forces in addition to X, Y, Z acting at every instant in a direction at right angles to the direction of the body's motion. Let X', Y', Z' be these forces, then they satisfy the condition

$$X'dx+Y'dy+Z'dz=0,$$

and therefore v is still given by equation (3).

By the principles of the calculus of Variations we have

$$\delta\int v ds=\int\delta.vds=\int(ds.\delta v+v\delta ds)$$

$$ds=vdt;$$

$$\therefore\ ds.\delta v=dt.v\delta v=\tfrac{1}{2}dt.\delta v^2$$

$$=\frac{1}{M}(X\delta x+Y\delta y+Z\delta z)\,dt \text{ by equations (2) and (3)}$$

$$=dt\left\{\left(\frac{dv_1}{dt}-\frac{X'}{M}\right)\delta x+\left(\frac{dv_2}{dt}-\frac{Y'}{M}\right)\delta y+\left(\frac{dv_3}{dt}-\frac{Z'}{M}\right)\delta z\right\}$$

$$=dv_1.\delta x+dv_2.\delta y+dv_3.\delta z-\frac{1}{M}(X'\delta x+Y'\delta y+Z'\delta z)dt,$$

where v_1, v_2, v_3 are the resolved parts of the velocity in the directions of the axes:

$$\text{also}\quad ds^2=dx^2+dy^2+dz^2;$$

$$\therefore\ ds.\delta ds=dx.\delta dx+dy.\delta dy+dz.\delta dz;$$

$$\therefore\ v\delta ds=v_1\delta dx+v^2\delta dy+v_3\delta dz$$

$$=v_1d\delta x+v_2d\delta y+v_3d\delta z;$$

$$\therefore ds\,.\,\delta v + v\delta ds = dv_1\,.\,\delta x + dv_2\,.\,\delta y + dv_3\,.\,\delta z + v_1 d\delta x + v_2 d\delta y + v_3 d\delta z - \frac{1}{M}(X'\delta x + Y'\delta y + Z'\delta z)dt$$

$$= d(v_1\delta x + v_2\delta y + v_3\delta z) - \frac{1}{M}(X'\delta x + Y'\delta y + Z'\delta z)dt;$$

$$\therefore \int \delta\,.\,v ds = v_1\delta x + v_2\delta y + v_3\delta z - \frac{1}{M}\int (X'\delta x + Y'\delta y + Z'\delta z)dt + C,$$

which is the indefinite integral. Now, since the extreme points are supposed constant, δx, δy and δz are separately zero at each limit, and therefore the definite integral $\int \delta\,.\,v ds$ between those points

$$= -\frac{1}{M}\int (X'\delta x + Y'\delta y + Z'\delta z)dt$$

between the same points.

When $X' = 0$, $Y' = 0$, $Z' = 0$, this definite integral is zero, so that when the particle is resigned to the action of the forces X, Y, Z, $\delta \int v ds$ becomes zero, shewing that, in this case, $\int v ds$ is a maximum or minimum, and from the nature of the case it cannot be a maximum, therefore it must be a minimum.

Again, let a particle be restricted to move on a smooth surface; the equations of motion are

$$\left.\begin{aligned} M\frac{d^2x}{dt^2} &= X + R\cos\lambda, \\ M\frac{d^2y}{dt^2} &= Y + R\cos\mu, \\ M\frac{d^2z}{dt^2} &= Z + R\cos\nu. \end{aligned}\right\} \quad \ldots\ldots(1).$$

where $\cos\lambda$, $\cos\mu$, $\cos\nu$ are the direction-cosines of the normal.

Then, as before, we have

$$v^2 = V^2 + f(x, y, z) - f(a, b, c) \ldots\ldots(2)$$

where $Mdf(x, y, z) = 2(Xdx + Ydy + Zdz)$.

Let the integral $\int v ds$ be taken between limits corresponding to two fixed points in the path of the body; then, in this case also, the integral is less than the corresponding integral would be if the particle were compelled to describe any other path on the surface by forces X', Y', Z' in addition to X, Y, Z the direction of whose resultant was always at right angles to the direction of the body's motion. We have, as before,

$$\delta \int v ds = \int \delta\,.\,v ds = \int (ds\,.\,\delta v + v\delta ds),$$

$$ds = v dt;$$

$$\therefore\ ds\,.\,\delta v = dt\,.\,v\delta v = \tfrac{1}{2}dt\,.\,\delta v^2$$

$$= \frac{1}{M}(X\delta x + Y\delta y + Z\delta v)\,dt \text{ from equation (2)}$$

$$= \left\{\left(\frac{dv_1}{dt} - \frac{X' + R\cos\lambda}{M}\right)\delta x + \left(\frac{dv_2}{dt} - \frac{Y' + R\cos\mu}{M}\right)\delta y + \left(\frac{dv_3}{dt} - \frac{Z' + R\cos\nu}{M}\right)\delta z\right\} dt$$

$$= dv_1\,.\,\delta x + dv_2\,.\,\delta y + dv\,.\,\delta z - \frac{1}{M}(X'\delta x + Y'\delta y + Z'\delta z)\,dt$$

since $\cos\lambda\,.\,\delta x + \cos\mu\,.\,\delta y + \cos\nu\,.\,\delta z = 0$, the variation indicated by δx, δy, δz being along the surface.

And, as before, $v\delta ds = v_1 d\delta x + v_2 d\delta y + v_3 d\delta z$,

so that $\delta\,.\,vds = d(v_1\delta x + v_2\delta y + v_3\delta z) - \frac{1}{M}(X'\delta x + Y'\delta y + Z'\delta z)\,dt$,

from which the conclusion follows as in the preceding case.

CHAPTER V.

75. HAVING described the methods of determining the motion of a particle when acted on by forces either constant or depending only on the position and motion of the particle under consideration, we will now proceed to another extensive class of problems, which can be solved (except so far as our analysis is deficient) by the aid of the same principles. This class consists of the cases of the motion of any number of particles under the action of their mutual attractions. There is one condition to which we shall consider these attractions always subject; namely, that when two particles A and B attract each other, either alone or constituting two out of a system of many, the actual force of attraction exerted by A on B is equal to that exerted by B on A, and these forces act in the line joining the two particles, and in opposite directions. This is the case in all attractions in nature with which we are acquainted, and is always assumed to hold whenever two particles are said to attract each other. It is in general assumed without any formal statement, though it is possible to conceive that it should not be the case. Under the term attraction is included both attraction and repulsion.

In treating this class of problems we shall not enter into the details of the solution of any of them, but shall confine our attention to finding the equations of motion, and indicating the different ways in which the solutions are usually effected. We shall also prove certain properties which are common to the motions of all systems of this class.

We in every case suppose the force of attraction to be a function of the masses of the particles and of the distance between them, and of nothing else.

76. First, let us consider the case of two particles moving in a straight line under their mutual attraction.

Let m, m' be the masses of the particles; P the force of attraction between them, estimated so as to be positive when the force is attractive, and negative when it is repulsive; x, x' their respective distances from some fixed point in their line of motion, at a time t from some fixed epoch. Then P will be a function of m, m', and $x - x'$.

Let x be supposed greater than x', then the equations of motion will be

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -P \\ m'\frac{d^2x'}{dt^2} &= P \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Adding these equations, we have

$$m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} = 0; \quad \ldots\ldots\ldots\ldots (2)$$

also, dividing by m and m' respectively and subtracting, we have

$$\frac{d^2(x-x')}{dt^2} = -P\left(\frac{1}{m} + \frac{1}{m'}\right) \ldots\ldots\ldots\ldots (3)$$

Since x and x' enter P only in the form $x - x'$ equation (3) is of the same form as the equation for the motion of one particle only, and may be integrated by the same methods. The integration of equation (2) can be easily effected.

The integrals of equations (2) and (3) will be two relations between x, x', t and constant quantities, from which x and x' may be found separately as functions of t, and the absolute motion of each of the particles determined. The constants introduced in integration may be determined, as before, by the positions of the particles and the circumstances of the motion at some known instant.

This is the most direct though not the most convenient method of obtaining the solution of the problem from equations (1).

Let $\bar{x}$ be the distance of the center of gravity of the two particles from the fixed point, at the time t. Then

$$(m + m')\bar{x} = mx + m'x' \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Differentiating this equation twice, we have

$$(m + m')\frac{d\bar{x}}{dt} = m\frac{dx}{dt} + m'\frac{dx'}{dt}, \quad \ldots\ldots\ldots\ldots (5)$$

$$\text{and } (m + m')\frac{d^2\bar{x}}{dt^2} = m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} \ldots\ldots\ldots\ldots\ldots (6)$$

Therefore, from equation (2) we have

$$\frac{d^2\bar{x}}{dt^2} = 0 \ldots\ldots\ldots\ldots\ldots (7)$$

$$\text{From which } \frac{d\bar{x}}{dt} = \text{constant.}$$

Now at some particular instant let the velocities of the two particles be v and v' respectively, then from equation (5)

$$(m+m')\frac{d\bar{x}}{dt} = mv + m'v',$$

and therefore, $\frac{d\bar{x}}{dt} = \frac{mv+m'v'}{m+m'} = \bar{v}$ suppose.

We have obtained the following result. If a point move so as to coincide at every instant with the center of gravity of the two particles m and m', it will advance with a uniform velocity $\bar{v}$; or, in other words, the center of gravity moves with a uniform velocity $\bar{v}$.

This property, which has been shewn to hold in this particular case, will be proved to hold in a great many descriptions of motion. It suggests another method of solving the problem under consideration.

77. This method is to determine the motions of the particles relatively to their center of gravity considered as a fixed point. If the relative position and velocity at any time is known, the absolute position and velocity can be immediately determined from it.

Let then r and r' be the distances of the particles from their common center of gravity; then we have

$$r = \frac{m'(x-x')}{m+m'}, \quad r' = \frac{m(x-x')}{m+m'},$$

$$\text{or, } (x-x') = \frac{m+m'}{m'}r = \frac{m+m'}{m}r'.$$

We may, therefore, put equation (3) into the two following forms:

$$\frac{d^2r}{dt^2} = -\frac{P}{m},$$

$$\frac{d^2r'}{dt^2} = -\frac{P}{m'};$$

and since P is a function of $x-x'$ and therefore of either r or r', these two equations may be integrated, and the motion relatively to the center of gravity determined.

In determining the constant after the first integration of the equation

$$\frac{d^2r}{dt^2} = -\frac{P}{m},$$

we have to substitute for $\frac{dr}{dt}$ its value at some known time, or in

some known position. It must be borne in mind that this value is not the velocity of the particle m at that time, but its velocity relatively to the center of gravity: that is, it is $v - \bar{v}$, not v.

Similarly, we should have to substitute for $\frac{dr'}{dt}$ the quantity $\bar{v} - v'$, since it is estimated in a direction opposite to that of v'.

This method is sometimes stated as follows. When two particles move in a straight line under their mutual attractions, the motion relatively to the center of gravity may be found thus: impress on each of the particles a velocity equal to that of the center of gravity, and in an opposite direction; this will reduce the center of gravity to rest, and not affect the motion relatively to it; take now the center of gravity as origin and determine the motion by the ordinary equations for rectilinear motion.

78. The integration of equation (3) gives the motion of the two particles relatively to each other. This is also sometimes stated as an independent method, as follows: to find the motion of a particle m relatively to a particle m' when they move in a straight line under their mutual attractions, impress on the particle m' a force equal to the force which m exerts on it, and in the opposite direction; there will then be no force acting on m'; also, in order that the relative motion may not be affected, impress on m a force whose accelerating force is equal to, and in the same direction as, that of the force impressed on m'; also, impress on each particle a velocity equal and opposite to the velocity of m' at some instant: m' will then remain at rest, and the motion of m relatively to it may be found by the ordinary equations of rectilinear motion.

It is easily shewn that this method will lead to equation (3); thus let r be the distance of the particles; the force which m exerts on m' is P, therefore the force impressed on m' is $-P$, and its accelerating force is $-\frac{P}{m'}$; therefore the force which must be impressed on m to have this accelerating force is $-\frac{mP}{m'}$, and the force that does act on it is $-P$; therefore the equation for its motion becomes

$$m\frac{d^2r}{dt^2} = -P - \frac{mP}{m'},$$

$$\text{or, } \frac{d^2r}{dt^2} = -P\left(\frac{1}{m} + \frac{1}{m'}\right),$$

which is the same as equation (3), with r in the place of $x - x'$.

Though in this case these two last methods are of little importance, from the simplicity of the direct method of solution, they will be seen to apply in other more general cases, where the direct solution would be very complicated.

79. There is another method of combining the original equations:

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -P \\ m'\frac{d^2x'}{dt^2} &= P \end{aligned}\right\}, \quad \ldots\ldots\ldots(1)$$

which is sometimes of use; we have

$$2m\frac{d^2x}{dt^2}\frac{dx}{dt} = -2P\frac{dx}{dt},$$

$$2m'\frac{d^2x'}{dt^2}\frac{dx'}{dt} = 2P\frac{dx'}{dt},$$

and adding these we have

$$2\left\{m\frac{d^2x}{dt^2}\frac{dx}{dt} + m'\frac{d^2x'}{dt^2}\frac{dx'}{dt}\right\} = -2P\frac{d(x-x')}{dt},$$

from which we have

$$m\left(\frac{dx}{dt}\right)^2 + m'\left(\frac{dx'}{dt}\right)^2 = C - 2\int P\,d(x-x'),$$

$$\text{or, } mv^2 + m'v'^2 = C - 2\int P\,d(x-x').$$

Since P is a function of $x - x'$ this integration can generally be effected.

80. Next, let there be any number of particles moving in a straight line under their mutual attractions.

Let $m_1, m_2, \ldots\ldots m_n$ be the masses of the particles.

$x_1, x_2, \ldots\ldots x_n$ their distances from some fixed point in their line of motion at time t.

${}_1P_2$ the force of attraction of m_2 on m_1, estimated as positive when tending to increase x_1, with a similar notation for the other forces, then the equations of motion will be

$$\left.\begin{aligned} m_1\frac{d^2x_1}{dt^2} &= {}_1P_2 + {}_1P_3 + \ldots\ldots + {}_1P_n \\ m_2\frac{d^2x_2}{dt^2} &= {}_2P_1 + {}_2P_3 + \ldots\ldots + {}_2P_n \\ \ldots\ldots &= \ldots\ldots\ldots\ldots\ldots \\ m_n\frac{d^2x_n}{dt^2} &= {}_nP_1 + {}_nP_2 + \ldots\ldots + {}_nP_{n-1} \end{aligned}\right\}, \ldots (1)$$

$$\left.\begin{aligned} \text{where } {}_1P_2 + {}_3P_1 &= 0 \\ {}_1P_3 + {}_2P_1 &= 0 \\ \ldots\ldots &= 0 \end{aligned}\right\} \ldots\ldots\ldots (2);$$

and ${}_1P_2$ is a function of $x_2 - x_1$, and the other forces are in like manner functions of the distances of the particles to which they correspond.

The solution of these n equations would determine the n quantities $x_1, x_2, \ldots x_n$ as functions of t, and thus completely determine the motion of each of the n particles.

If we add together equations (1), and bear in mind equations (2), we have

$$m_1 \frac{d^2x_1}{dt^2} + m_2 \frac{d^2x_2}{dt^2} + \ldots\ldots + m_n \frac{d^2x_n}{dt^2} = 0 \ldots (3).$$

Now if $\bar{x}$ be the distance of the center of gravity of all the particles from the fixed point, we have

$$(m_1 + m_2 + \ldots + m_n)\,\bar{x} = m_1x_1 + m_2x_2 + \ldots + m_nx_n;$$

$$\therefore\ (m_1 + m_2 + \ldots + m_n)\frac{d^2\bar{x}}{dt^2} = m_1\frac{d^2x_1}{dt^2} + m_2\frac{d^2x_2}{dt^2} + \ldots + m_n\frac{d^2x_n}{dt^2} = 0;$$

$$\therefore\ \bar{v} = \frac{d\bar{x}}{dt} = \text{const.}$$

or the motion of the center of gravity is uniform.

We shall hereafter see that this is the case when particles move in any way under their mutual attractions.

Again, from equations (1) and (2)

$$2\left\{m_1\frac{d^2x_1}{dt^2}\frac{dx_1}{dt} + m_2\frac{d^2x_2}{dt^2}\frac{dx_2}{dt} + \ldots + m_n\frac{d^2x_n}{dt^2}\frac{dx_n}{dt}\right\}$$

$$= 2\,{}_1P_2\frac{d(x_1 - x_2)}{dt^2} + 2\,{}_1P_3\frac{d(x_1 - x_3)}{dt^2} + \&c.$$

and integrating

$$m_1\left(\frac{dx_1}{dt}\right)^2 + m_2\left(\frac{dx_2}{dt}\right)^2 + \ldots + m_n\left(\frac{dx_n}{dt}\right)^2 = 2\int {}_1P_2\,d(x_1 - x_2) + \&c. + C,$$

or, as it is generally written,

$$\Sigma mv^2 = C + 2\Sigma\int {}_1P_2\,d(x_1 - x_2).$$

These integrations can generally be effected.

We have thus obtained two integrals of the n equations (1); the methods for obtaining others will depend on the forms of the functions P, and must be left to the ingenuity of the analyst.

81. Let us now take the case of two bodies moving in one plane, but not in the same straight line. This is possible, for if at any instant the directions of their motion lie in one plane, they will always lie in that plane, since no force acts to move them out of it.

Let m, m' be the masses of the two particles x, x' and y, y' their co-ordinates at time t; r their distance; P the force of attraction between them; then the equations of motion are

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -P\frac{x-x'}{r} \\ m\frac{d^2y}{dt^2} &= -P\frac{y-y'}{r} \\ m'\frac{d^2x'}{dt^2} &= -P\frac{x'-x}{r} \\ m'\frac{d^2y'}{dt^2} &= -P\frac{y'-y}{r} \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots (1)$$

with the geometrical relation

$$r^2 = (x-x')^2 + (y-y')^2, \quad \ldots\ldots\ldots (2)$$

where P is a function of r and constant quantities.

From these equations we have

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} &= 0, \\ m\frac{d^2y}{dt^2} + m'\frac{d^2y'}{dt^2} &= 0, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3)$$

from which it appears as in the preceding cases that $\frac{d\bar{x}}{dt}$ and $\frac{d\bar{y}}{dt}$, the resolved parts of the velocity of the center of gravity of the two bodies, are constant; and therefore the center of gravity moves in a straight line with uniform velocity.

Again we have

$$2m\left(\frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{d^2y}{dt^2}\frac{dy}{dt}\right) + 2m'\left(\frac{d^2x'}{dt^2}\frac{dx'}{dt} + \frac{d^2y'}{dt^2}\frac{dy'}{dt}\right)$$

$$= -\frac{2P}{r}\left\{(x-x')\frac{d(x-x')}{dt} + (y-y')\frac{d(y-y')}{dt}\right\}$$

$$= -2P\frac{dr}{dt};$$

$$\therefore\ m\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} + m'\left\{\left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2\right\} = C - 2\int P\,dr,$$

$$\text{or}\quad mv^2 + m'v'^2 = C - 2\int P\,dr \ldots\ldots\ldots (4)$$

This is another integral of equations (1). We may obtain another as follows:

$$m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)+m'\left(x'\frac{d^2y'}{dt^2}-y'\frac{d^2x'}{dt^2}\right)=0.$$

From which we have

$$m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)+m'\left(x'\frac{dy'}{dt}-y'\frac{dx'}{dt}\right)=h\ldots\ldots(5)$$

(4) and (5), together with the equations

$$\left.\begin{aligned} m\frac{dx}{dt}+m'\frac{dx'}{dt}&=c,\\ m\frac{dy}{dt}+m'\frac{dy'}{dt}&=c',\end{aligned}\right\}\ldots\ldots\ldots(6)$$

which we derive from (3), are the first integrals of equations (1). If we can solve these the absolute motions of the particles m and m' will be completely determined.

Generally, however, it is not the absolute motions of the two particles that are required, but their motions relatively to their center of gravity, or to each other. First, then, to find the motion of m relatively to m', let x_1y_1 be the co-ordinates of m considering m' as origin, that is, let

$$x_1=x-x',\quad y_1=y-y'.$$

Then we have, from equations (1),

$$\left.\begin{aligned} \frac{d^2(x-x')}{dt^2}&=-P\frac{m+m'}{mm'}\frac{x-x'}{r},\\ \text{or }\frac{d^2x_1}{dt^2}&=-P\frac{m+m'}{mm'}\frac{x_1}{r}\\ \text{and }\frac{d^2y_1}{dt^2}&=-P\frac{m+m'}{mm'}\frac{y_1}{r}\end{aligned}\right\}\ldots\ldots\ldots(7)$$

These equations are the same as those for the motion of a particle round a fixed center of force: all the properties therefore which are proved to be true in that case, are true also for the motion of one of two bodies relatively to the other; and the same methods of solution apply in this case as in that. Writing P' for $P\frac{m+m'}{mm'}$, we have the two polar equations of motion

$$\left.\begin{aligned} \frac{d^2u}{d\theta^2}+u-\frac{P'}{h^2u^2}&=0,\\ \frac{d\theta}{dt}&=hu^2.\end{aligned}\right\}\ldots\ldots\ldots(8)$$

These will be the equations from which the solution will generally be best obtained. This method may be enunciated in words thus.

To find the motion of m relatively to m', apply to m' a force equal and in the opposite direction to that exerted by m, then there will be no force acting on m': in order that the relative motion may not be disturbed apply to m a force such that its accelerating force shall be equal to that of the force applied to m': also apply to each of the bodies a velocity equal and in the opposite direction to that with which m' is moving, m' will then be reduced to rest and the relative motion of m will be the absolute motion.

A little consideration will shew at once that this, which we have given as the interpretation of equations (7), might have been assumed *a priori* as true. The motion of m' relatively to m will be exactly similar to this.

To find the motion relatively to their center of gravity.

Let ξ, η be the co-ordinates of m relatively to their center of gravity, ρ the distance of m from it; then we have

$$\rho = \frac{m'}{m+m'} r,$$

$$\xi = \frac{m'}{m+m'}(x-x'), \quad \eta = \frac{m'}{m+m'}(y-y');$$

and equations (7) become

$$\left.\begin{aligned} m\frac{d^2\xi}{dt^2} &= -P\frac{\xi}{\rho}, \\ m\frac{d^2\eta}{dt^2} &= -P\frac{\eta}{\rho}; \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (9)$$

the equations of motion relatively to the center of gravity.

P, which is a function of r, will be a function therefore of ρ. These are the equations of motion which we should have had if the body had been moving round the center of gravity fixed, acted on by the same force. The only points of difference will be in the determination of the constants introduced in integration. We may enunciate the method therefore as follows.

Impress on the system a velocity equal and opposite to that with which the center of gravity is moving; the center of gravity will then be reduced to rest, and the motion about it may be determined by the ordinary methods for determining the motion about a fixed center, and all the conclusions arrived at in motion of that kind will be true also in this.

If the absolute motion be required we can compound the relative motion thus found with the motion of the center of gravity.

82. Let us now consider the case where the motions of the two particles are not in one plane. Using a notation similar to that we have already adopted, the equations of motion are

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -P\frac{x-x'}{r}, & m'\frac{d^2x'}{dt^2} &= -P\frac{x'-x}{r} \\ m\frac{d^2y}{dt^2} &= -P\frac{y-y'}{r}, & m'\frac{d^2y'}{dt^2} &= -P\frac{y'-y}{r} \\ m\frac{d^2z}{dt^2} &= -P\frac{z-z'}{r}, & m'\frac{d^2z'}{dt^2} &= -P\frac{z'-z}{r} \end{aligned}\right\} \ldots (1),$$

From these equations we deduce as in the preceding cases, the equations

$$\frac{d\bar{x}}{dt} = c_1, \quad \frac{d\bar{y}}{dt} = c_2, \quad \frac{d\bar{z}}{dt} = c_3 \ldots\ldots\ldots (2)$$

which shew that the center of gravity moves uniformly in a straight line.

We may also obtain as before the equation

$$mv^2 + m'v'^2 = C - 2\int P dr \ldots\ldots\ldots\ldots\ldots (3)$$

bearing in mind that

$$r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2.$$

And we shall also have, in place of equation (5) of the preceding case, the three equations

$$\left.\begin{aligned} m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) + m'\left(y'\frac{dz'}{dt} - z'\frac{dy'}{dt}\right) &= h_1 \\ m\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) + m'\left(z'\frac{dx'}{dt} - x'\frac{dz'}{dt}\right) &= h_2 \\ m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) + m'\left(x'\frac{dy'}{dt} - y'\frac{dx'}{dt}\right) &= h_3 \end{aligned}\right\} \ldots (4)$$

Equations (2), (3) and (4), are the first integrals of equations (1), and when they can be again integrated, give the direct solution of the problem.

We might deduce the equations giving the motion of m relatively to m' or of either relatively to their common center of gravity. It is also easily seen that the principles before stated for reducing the motion of one relatively to the other, or to their center of gravity to the case of motion about a fixed center will apply in this case also. Hence it follows, that in whatever manner two bodies move under their mutual attractions, the motion of one relatively to the other, as also the motions of the two relatively to their center of gravity will be in one plane. This plane however will not coincide

with the direction of motion of the center of gravity, or of that body to which the motion of the other is referred.

When we have by one of the principles above stated reduced the motion to that about a fixed center, it will become only a particular case of the motion of one particle, which has already been fully considered.

83. Let us now consider the case of any number of particles moving under their mutual attractions.

Let $m_1, m_2, \ldots m_n$ be the masses of the several particles; x_1, y_1, z_1; $x_2, y_2, z_2; \ldots x_n, y_n, z_n$ their co-ordinates at time t; ${}_1P_2$ the absolute force of attraction between m_1 and m_2; with similar expressions for the other forces; ${}_1r_2$ the distance between m_1 and m_2 with similar expressions for the other distances.

The equations of motion are

$$m_1 \frac{d^2x_1}{dt^2} = -{}_1P_2 \frac{x_1 - x_2}{{}_1r_2} - {}_1P_3 \frac{x_1 - x_3}{{}_1r_3} - \ldots - {}_1P_n \frac{x_1 - x_n}{{}_1r_n},$$

$$m_1 \frac{d^2y_1}{dt^2} = -{}_1P_2 \frac{y_1 - y_2}{{}_1r_2} - {}_1P_3 \frac{y_1 - y_3}{{}_1r_3} - \ldots - {}_1P_n \frac{y_1 - y_n}{{}_1r_n},$$

$$m_1 \frac{d^2z_1}{dt^2} = -{}_1P_2 \frac{z_1 - z_2}{{}_1r_2} - \ldots\ldots\ldots\ldots - {}_1P_n \frac{z_1 - z_n}{{}_1r_n},$$

with similar equations for all the particles.

Together with the geometrical conditions

$${}_1r_2^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

and similar equations for the other distances.

From these equations, we have

$$m_1 \frac{d^2x_1}{dt^2} + m_2 \frac{d^2x_2}{dt^2} + \ldots + m_n \frac{d^2x_n}{dt^2} = 0,$$

$$m_1 \frac{d^2y_1}{dt^2} + \ldots\ldots\ldots\ldots + m_n \frac{d^2y_n}{dt^2} = 0,$$

$$m^1 \frac{d^2z_1}{dt^2} + \ldots\ldots\ldots\ldots + m_n \frac{d^2z_n}{dt^2} = 0,$$

from which, if $\bar{x}, \bar{y}, \bar{z}$ are the co-ordinates of the center of gravity,

$$\frac{d\bar{x}}{dt} = c_1, \quad \frac{d\bar{y}}{dt} = c^2, \quad \frac{d\bar{z}}{dt} = c_3,$$

or the center of gravity moves in a straight line with uniform velocity.

This property which we have seen to hold in all the particular cases before considered, and have now proved to hold in the general case of the motion of any number of particles under their mutual attractions, is a particular case of a much more general principle, called "the principle of the conservation of the motion of the center of gravity," which will be stated and proved hereafter.

84. Again, multiplying each second differential coefficient by twice the corresponding first differential coefficient and adding all together, and bearing in mind the geometrical relations which give

$$2\,{}_1r_2\frac{d\,{}_1r_2}{dt} = 2(x_1-x_2)\left(\frac{dx_1}{dt}-\frac{dx_2}{dt}\right)+2(y_1-y_2)\left(\frac{dy_1}{dt}-\frac{dy_2}{dt}\right)$$
$$+2(z_1-z_2)\left(\frac{dz_1}{dt}-\frac{dz_2}{dt}\right),$$

we have

$$2m_1\left\{\frac{d^2x_1}{dt^2}\frac{dx_1}{dt}+\frac{d^2y_1}{dt^2}\frac{dy_1}{dt}+\frac{d^2z_1}{dt^2}\frac{dz_1}{dt}\right\}+\&c. = -2\,{}_1P_2\frac{d\,{}_1r_2}{dt}-\&c.$$

And we observe on the right-hand side that ${}_1P_2$ is a function of ${}_1r_2$ and constant quantities, and, in like manner, the other forces of the corresponding distances; therefore integrating, we have

$$m\left\{\left(\frac{dx_1}{dt}\right)^2+\left(\frac{dy_1}{dt}\right)^2+\left(\frac{dz_1}{dt}\right)^2\right\}+\&c. = C-2\int {}_1P_2\,d\,{}_1r_2-\&c.$$

or, as it may be written,

$$\Sigma(mv^2) = C-2\Sigma\int P\,dr.$$

We have, in particular cases already considered, obtained equations corresponding to this.

The name "vis viva" has been given to the product mv^2, $\Sigma(mv^2)$ which is the sum of the "vires vivæ" of the several particles of the system is called the "vis viva" of the system.

We will now ascertain what the last formed equation assures us of.

Let $2\int {}_1P_2\,d\,{}_1r_2$ be ${}_1f_2({}_1r_2)$,

then it becomes $\Sigma(mv^2) = C-\Sigma\{f(r)\}$.

Now let V_1, $V_2\ldots V_n$ be the velocities of the several particles at the time when their mutual distances are ${}_1a_2$, ${}_1a_3$, ${}_2a_3$, &c. then we have

$$\Sigma(mV^2) = C-\Sigma\{f(a)\},$$

from which we have

$$\Sigma(mv^2)-\Sigma(mV^2) = \Sigma f(a)-\Sigma f(r).$$

This equation shews that the change in the vis viva of the system depends only on the relative positions of the particles at the two instants when it is estimated, and is independent of the directions in which they are moving or the paths they have described.

85. Again, recurring to our equations of motion, it will be seen on examination that they give the following equations,

$$m_1\left(y_1\frac{d^2z_1}{dt^2}-z_1\frac{d^2y_1}{dt^2}\right)+m_2\left(y_2\frac{d^2z_2}{dt^2}-z_2\frac{d^2y_2}{dt^2}\right)+\&c.=0,$$

$$m_1\left(z_1\frac{d^2x_1}{dt^2}-x_1\frac{d^2z_1}{dt^2}\right)+m_2\left(z_2\frac{d^2x_2}{dt^2}-x_2\frac{d^2z_2}{dt^2}\right)+\&c.=0,$$

$$m_1\left(x_1\frac{d^2y_1}{dt^2}-y_1\frac{d^2x_1}{dt^2}\right)+m_2\left(x_2\frac{d^2y_2}{dt^2}-y_2\frac{d^2x_2}{dt^2}\right)+\&c.=0.$$

Integrating these equations, we have

$$m_1\left(y_1\frac{dz_1}{dt}-z_1\frac{dy_1}{dt}\right)+m_2\left(y_2\frac{dz_2}{dt}-z_2\frac{dy_2}{dt}\right)+\&c.=h_1,$$

$$m_1\left(z_1\frac{dx_1}{dt}-x_1\frac{dz_1}{dt}\right)+m_2\left(z_2\frac{dx_2}{dt}-x_2\frac{dz_2}{dt}\right)+\&c.=h_2,$$

$$m_1\left(x_1\frac{dy_1}{dt}-y_1\frac{dx_1}{dt}\right)+m_2\left(x_2\frac{dy_2}{dt}-y_2\frac{dx_2}{dt}\right)+\&c.=h_3,$$

or, as we may write them conveniently

$$\Sigma m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)=h_1,$$

$$\Sigma m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right)=h_2,$$

$$\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=h_3,$$

where h_1, h_2, h_3 are constant.

Let us now consider what these equations express.

If we conceive a line drawn from the origin of co-ordinates to a particle m to generate an area by its motion, and if A be this area at time t, and A_1, A_2, A_3 its projections on the three co-ordinate planes, or, which amounts to the same thing, if we conceive that A_1, A_2, A_3, are the areas described on the co-ordinate planes by the projections of this imaginary line, then it is known from geometry, that

$$y\frac{dz}{dt}-z\frac{dy}{dt}=2\frac{dA_1}{dt},$$

$$z\frac{dx}{dt}-x\frac{dz}{dt}=2\frac{dA_2}{dt},$$

$$x\frac{dy}{dt}-y\frac{dx}{dt}=2\frac{dA_3}{dt}.$$

These three equations then give

$$2\Sigma\left(m\frac{dA_1}{dt}\right)=h_1,$$

$$2\Sigma\left(m\frac{dA_2}{dt}\right)=h_2,$$

$$2\Sigma\left(m\frac{dA_3}{dt}\right)=h_3.$$

And integrating them we have

$$2\Sigma(mA_1)=h_1t,$$

$$2\Sigma(mA_2)=h_2t,$$

$$2\Sigma(mA_3)=h_3t,$$

making the constants which are introduced by integration each zero, which amounts to supposing that the description of areas commences at the instant from which t is measured.

These equations shew that if the mass of each particle of the system be multiplied by the area described by the projection on each of the co-ordinate planes of the line joining it with the origin, the sum of all these products will be proportional to the time of describing them, whatever point be assumed as origin.

This is a case of a general principle which will be proved hereafter, called the "Principle of the Conservation of Areas."

86. Let λ, μ, ν, be the direction-cosines of the normal to a plane, then the projection of the area A on this plane is

$$\lambda A_1+\mu A_2+\nu A_3,$$

and a similar expression will give the projection of the area for every particle.

Therefore, the sum of the products of each mass into the projection on this plane of the area described by it is

$$\lambda\Sigma(mA_1)+\mu\Sigma(mA_2)+\nu\Sigma(mA_3).$$

Now, to find the plane on which this sum is a maximum at any time we must make this expression a maximum by the variation of λ, μ, and ν, subject to the single condition

$$\lambda^2+\mu^2+\nu^2=1.$$

We have then

$$\Sigma(mA_1)\,\delta\lambda+\Sigma(mA_2)\,\delta\mu+\Sigma(mA_3)\,\delta\nu=0,$$

$$\lambda\,\delta\lambda+\mu\,\delta\mu+\nu\,\delta\nu=0;$$

therefore, using the indeterminate multiplier B we must have

$$\Sigma(mA_1) + B\lambda = 0,$$
$$\Sigma(mA_2) + B\mu = 0,$$
$$\Sigma(mA_3) + B\nu = 0;$$

and therefore $\dfrac{\lambda}{\Sigma(mA_1)} = \dfrac{\mu}{\Sigma(mA_2)} = \dfrac{\nu}{\Sigma(mA_3)}$,

or $\dfrac{\lambda}{h_1 t} = \dfrac{\mu}{h_2 t} = \dfrac{\nu}{h_3 t}$;

and therefore, if $h^2 = h_1^2 + h_2^2 + h_3^2$, we have

$$\lambda = \frac{h_1}{h}, \quad \mu = \frac{h_2}{h}, \quad \nu = \frac{h_3}{h}.$$

That is, the position of the plane is constant during the motion; it is called the plane of maximum areas, or the "Invariable plane."

We may remark that the position in space of the plane has not been proved constant, but merely the direction of its normal; all planes parallel to this would have the same property.

If at any instant we know the positions, magnitudes, velocities, and directions of motion of the several particles of such a system as we have been considering, we can find the position of this invariable plane, with reference to co-ordinate axes arbitrarily chosen; and if at any future time we take other co-ordinate axes arbitrarily chosen, we can also determine with reference to them the position of the invariable plane, and we know that these two positions must be absolutely the same, as far as regards direction, whatever the co-ordinates by which they are determined. Thus we see that the system itself furnishes a plane of reference which is invariable in direction, and can always be determined from the state of the system at any instant.

The preceding are the principal general properties of a system of particles moving under their mutual actions; the actual determination of the motion of each particle in anything like the general case far surpasses our powers of analysis. The solution of a number of simple cases is the only means of acquiring a familiarity with the treatment of problems of this sort.

87. We have hitherto supposed the system to be acted on by no forces but those which arise from the mutual attractions of its several particles; in addition to these, all or any of the particles of the system may be acted on by forces arising from causes extraneous to the system, as, for instance, the attraction of some body not

forming a part of the system; or by forces arising from geometrical constraints, such as, that some of them move in tubes or on surfaces.

In these cases if the forces are given, they will enter our equations as known quantities, and we shall still have as many equations as we have co-ordinates to determine.

If the forces are not given explicitly, but are known to arise from the particles being compelled to satisfy certain geometrical conditions, they will then enter the equations as unknown quantities; and, corresponding to each unknown force, there will be an equation expressing the geometrical condition from which it arises; so that the whole number of equations will in every case be the same as the whole number of unknown quantities, and will thus be sufficient to express every such unknown quantity as a function of t.

We shall not, however, enter upon any of these cases here, as our object is not to obtain solutions of particular problems, but to explain the principles by which those solutions are to be obtained, and to investigate certain properties which belong to extensive classes of such problems.

CHAPTER VI.

88. In the preceding pages the motion of a single material particle, and also of a system of several particles has been considered, when the particles were acted on by various forces and subject to different conditions; it remains to investigate the methods of determining the motion of bodies of finite magnitude.

Now, when a body of finite magnitude is in motion, unknown forces act on every particle of it arising from the action of every other particle. The determination of these forces from the equations expressing the conditions to which the system is subject presents insuperable difficulties. The consideration of these forces is avoided by making use of a principle first stated by D'Alembert, which we will now proceed to explain.

When a material system is in motion under the action of forces arising from causes external to the system, these external forces are called "impressed forces;" and those of them which act on any one particle of the system are called the impressed forces on that particle.

Every particle besides being acted on by the impressed forces, is also acted on by forces arising from its connexion with the other parts of the system; these are called "internal forces."

The whole force acting on any particle is the resultant of the impressed and internal forces on that particle, and is called the "effective" force on that particle.

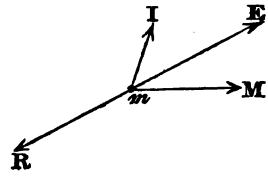
Conceive an equal isolated particle to be moving in the same direction and with the same velocity as the proposed particle, and to be acted on by a single force equal to this resultant and in the same direction; the two particles will be in the same state, one being acted on by a number of forces, the other by a single force which is their resultant; the motions of the two particles will therefore be the same. Hence then we sometimes have the following definition of effective force.

That force which acting alone on any particle of a system supposed isolated would cause it to move as it does when it forms part of the system; that is, which would generate in the small time δt the same change in velocity and direction of motion that takes place when the particle forms part of the system, is called the "effective" force on that particle.

Now D'Alembert's principle asserts that if at any instant we apply to every particle of a system forces equal and opposite to the effective forces on that particle, the system, if it were at rest in the position which it occupies at that instant, would remain in equilibrium under the action of these together with the impressed forces.

We have enunciated this principle in the form in which it will be most convenient for obtaining equations of motion for a system. We will now explain the nature of the assertion contained in it.

Let m be a particle of the system; I the resultant impressed force acting on that particle; M the resultant of the internal forces on it; E the resultant of M and I, and therefore the effective force on the particle; R the reversed effective force, that is, a force equal to E acting in the opposite direction.



Then if the particle m were at rest, and the forces I, M, and R acted on it, they would keep it in equilibrium: and the same would be true of every particle.

If therefore the whole system were at rest, and each particle acted on by the impressed forces, the effective forces reversed, and the same internal forces that acted on it when the system was in motion, every particle would be separately in equilibrium, and therefore the whole system would be in equilibrium.

Instead of this suppose the system to be at rest, and each particle to be acted on by the impressed forces and reversed effective forces that correspond to it, and by such internal forces as naturally arise from this condition; then D'Alembert's Principle asserts that the system will in this case also be in equilibrium.

The Principle may also be considered in the following light, in which it possibly appears less arbitrary.

A system is in motion under the action of impressed forces and internal forces arising from the connexion of its parts. Let the impressed forces be replaced by the effective forces: in this case, since each particle will move under the action of its effective force alone consistently with the connexion of the parts of the system, the internal forces arising from this connexion will not be called into action, and the change of motion in the small time δt will be the same as under the action of the impressed forces.

Now, forces equal and opposite to the effective forces, acting together with the effective forces on the system at rest, keep it in equilibrium; will they keep it in equilibrium when they act together with the impressed forces? D'Alembert's Principle asserts that they

will. It states that a set of forces which balances one of two sets that produce the same change of motion in a system will also balance the other.

D'Alembert's Principle, like the laws of motion and the theory of gravitation, is proved by the agreement between the results of calculation and observation. We are not, however, able in this case to avail ourselves of Astronomical observation. The comparison has been principally made by means of a machine invented by Atwood, by which the motion of a system of bodies can be very accurately observed in a great variety of cases.

89. To form the equations of motion by the aid of this principle, we must express the effective force on each particle in terms of the co-ordinates of that particle: if m be the mass of a particle, x, y, z its co-ordinates, the resolved parts of this force at the time t will be $m\frac{d^2x}{dt^2}$, $m\frac{d^2y}{dt^2}$, and $m\frac{d^2z}{dt^2}$. We must now suppose the system to be at rest in the position it occupies at the time t, and apply to every particle a force equal and opposite to the effective force on it. The system will then be in equilibrium under the action of these and the impressed forces: and the equations expressing the conditions of equilibrium of the system will be the equations of motion.

These equations will be different for different systems of bodies. They will be different according as there is one body or more, as the bodies are elastic or inelastic, solid or fluid, rigid or flexible.

The equations expressing the conditions of equilibrium assume different forms in these different cases, and consequently the equations of motion obtained from them by the aid of D'Alembert's principle are of different forms.

90. We will now apply this principle in the form in which we have enunciated it to the solution of a problem; viz. to calculate the motion of a uniform cylinder with its axis horizontal, rolling down an inclined plane so rough as to prevent all sliding.

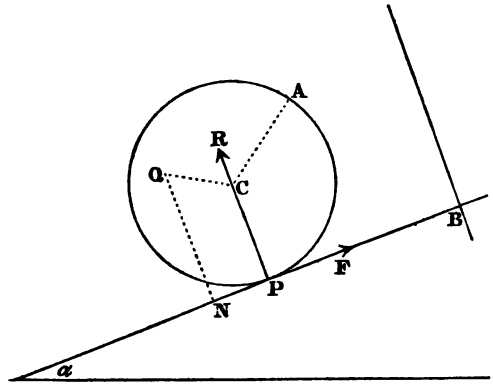
Let the figure represent a section of the cylinder and inclined plane; and let a be the radius of the cylinder, h its length, α the inclination of the plane to the horizon.

Let any point B in the plane be the origin, BP the axis of x, a normal to the plane at B the axis of y; Q an element of the cylinder, δm its mass, x, y, z its co-ordinates; then $BN = x$, $QN = y$.

Let $BP = \bar{x}$, $QC = r$, $QCA = \phi$.

$ACP = \theta$ the angle through which the cylinder has rolled.

Since z is constant $\frac{d^2z}{dt^2} = 0$, and the resolved parts of the effective



force on Q are $\delta m \frac{d^2x}{dt^2}$ and $\delta m \frac{d^2y}{dt^2}$, parallel to the axes of x and y; and the impressed force is $g\,\delta m$ vertically downwards. Also on the line of particles in contact with the plane there will be other impressed forces arising from the reaction of the plane: let the resolved parts of one of these parallel to PC and PB be R' and F'.

Now apply to every particle the reversed effective forces on that particle, and write down the equations of equilibrium: they are

$$\Sigma R' - \Sigma \delta m \frac{d^2y}{dt^2} - \Sigma \delta m g \cos\alpha = 0,$$

$$\Sigma F' + \Sigma \delta m \frac{d^2x}{dt^2} - \Sigma \delta m g \sin\alpha = 0.$$

$$\Sigma F' a + \Sigma \delta m (x-\bar{x}) \frac{d^2y}{dt^2} - \Sigma \delta m (y-a) \frac{d^2x}{dt^2} + \Sigma \delta m g \cos\alpha (x-\bar{x}) + \Sigma \delta m g \sin\alpha (y-a) = 0.$$

To these equations we must add the geometrical relations

$$x = \bar{x} - r\sin(\theta+\phi),\quad y = a - r\cos(\theta+\phi),$$

from which we obtain

$$\frac{d^2x}{dt^2} = \frac{d^2\bar{x}}{dt^2} - (x-\bar{x})\left(\frac{d\theta}{dt}\right)^2 + (y-a)\frac{d^2\theta}{dt^2},$$

$$\frac{d^2y}{dt^2} = \quad -(y-a)\left(\frac{d\theta}{dt}\right)^2 - (x-\bar{x})\frac{d^2\theta}{dt^2}.$$

Let $\Sigma R' = R$, $\Sigma F' = F$, $\Sigma \delta m = M$, then substituting these values in the equations we have obtained, and bearing in mind that $\bar{x}$, $\bar{y}$,

and θ refer to the whole cylinder, and that the center of gravity of the cylinder is in the axis, and therefore that

$$\Sigma \delta m (x - \bar{x}) = 0, \quad \Sigma \delta m (y - a) = 0,$$

we have
$$R - Mg \cos \alpha = 0 \ldots\ldots\ldots\ldots (1)$$

$$M \frac{d^2 \bar{x}}{dt^2} = Mg \sin \alpha - F \ldots\ldots\ldots (2)$$

$$\frac{d^2 \theta}{dt^2} \Sigma \delta m r^2 = Fa \ldots\ldots\ldots\ldots (3)$$

also
$$\bar{x} = a\theta + c \ldots\ldots\ldots\ldots (4).$$

Equation (1) determines the whole pressure on the plane, which is constant.

Also
$$\Sigma \delta m r^2 = \tfrac{1}{2} M a^2,$$

therefore equation (3) becomes

$$\frac{Ma}{2} \frac{d^2 \theta}{dt^2} = F.$$

Eliminating F between this equation and (2) and reducing by means of (4), we have

$$\frac{d^2 \theta}{dt^2} = \frac{2g \sin \alpha}{3a};$$

$$\therefore \frac{d\theta}{dt} = \frac{2g \sin \alpha \, . \, t}{3a} + C,$$

$$\theta = \frac{g \sin \alpha \, . \, t^2}{3a} + Ct + C',$$

$\frac{d\theta}{dt}$ is called the angular velocity of the cylinder.

C and C' may be determined from the values of θ and $\frac{d\theta}{dt}$ corresponding to any value of t; and the motion is then completely determined.

We see from the preceding example, that even in a simple case the equations given by the immediate application of the principle are complicated, and require considerable reduction before they can be integrated. In the case of rigid bodies, however, this reduction can be effected generally, as follows:

91. Let there be one rigid body, and let m_1 be the mass of a particle, x_1, y_1, z_1 its co-ordinates at time t, X_1 Y_1 Z_1 the resolved parts of the impressed forces that act on it. The resolved parts of the effective force on it are

$$m_1 \frac{d^2 x_1}{dt^2}, \quad m_1 \frac{d^2 y_1}{dt^2}, \quad m_1 \frac{d^2 z_1}{dt^2}.$$

If we apply these in the opposite directions, the resolved parts of the whole force on the particle will be

$$X_1 - m_1 \frac{d^2x_1}{dt^2}, \quad Y_1 - m_1 \frac{d^2y_1}{dt^2} \text{ and } Z_1 - m_1 \frac{d^2z_1}{dt^2};$$

and the body will be in equilibrium under the action of these and similar forces applied to every particle.

Hence then the six equations of equilibrium give

$$\left.\begin{aligned} \Sigma\left(X - m\frac{d^2x}{dt^2}\right) &= 0, \\ \Sigma\left(Y - m\frac{d^2y}{dt^2}\right) &= 0, \\ \Sigma\left(Z - m\frac{d^2z}{dt^2}\right) &= 0, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \text{(I)}$$

$$\left.\begin{aligned} \Sigma\left\{y\left(Z - m\frac{d^2z}{dt^2}\right) - z\left(Y - m\frac{d^2y}{dt^2}\right)\right\} &= 0, \\ \Sigma\left\{z\left(X - m\frac{d^2x}{dt^2}\right) - x\left(Z - m\frac{d^2z}{dt^2}\right)\right\} &= 0, \\ \Sigma\left\{x\left(Y - m\frac{d^2y}{dt^2}\right) - y\left(X - m\frac{d^2x}{dt^2}\right)\right\} &= 0. \end{aligned}\right\} \quad \ldots \text{(II)}$$

These are the six equations of motion for one rigid body, and when these equations are solved the motion of the body is completely determined.

92. The preceding equations contain the co-ordinates of every particle of the body, the use of them is greatly facilitated by the following reduction.

Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the center of gravity of the body at the time t; x', y', z' the co-ordinates of a particle of the body referred to axes whose origin is the center of gravity and which are parallel to the original ones: then

$$x = \bar{x} + x', \quad y = \bar{y} + y', \quad z = \bar{z} + z',$$

and therefore

$$\frac{d^2x}{dt^2} = \frac{d^2\bar{x}}{dt^2} + \frac{d^2x'}{dt^2},$$

$$\frac{d^2y}{dt^2} = \frac{d^2\bar{y}}{dt^2} + \frac{d^2y'}{dt^2},$$

$$\frac{d^2z}{dt^2} = \frac{d^2\bar{z}}{dt^2} + \frac{d^2z'}{dt^2}.$$

Substituting in equations (I), we have

$$\Sigma X - \Sigma m \frac{d^2\bar{x}}{dt^2} - \Sigma m \frac{d^2x'}{dt^2} = 0,$$

$$\Sigma Y - \Sigma m \frac{d^2\bar{y}}{dt^2} - \Sigma m \frac{d^2y'}{dt^2} = 0,$$

$$\Sigma Z - \Sigma m \frac{d^2\bar{z}}{dt^2} - \Sigma m \frac{d^2z'}{dt^2} = 0,$$

but since the origin of $x'\ y'\ z'$ is the center of gravity

$$\Sigma m x' = 0, \quad \Sigma m y' = 0, \quad \Sigma m z' = 0,$$

and therefore

$$\Sigma m \frac{d^2x'}{dt^2} = 0, \quad \Sigma m \frac{d^2y'}{dt^2} = 0, \quad \Sigma m \frac{d^2z'}{dt^2} = 0,$$

also $\bar{x}$, $\bar{y}$, $\bar{z}$ are the same for every particle of the body, therefore writing M for Σm, we have

$$\left.\begin{aligned} M\frac{d^2\bar{x}}{dt^2} &= \Sigma X, \\ M\frac{d^2\bar{y}}{dt^2} &= \Sigma Y, \\ M\frac{d^2\bar{z}}{dt^2} &= \Sigma Z. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \text{(III)}$$

These three equations are in the form in which they are most available for determining the motion of the body. They are the same as those for the motion of a particle whose mass is M acted on by the forces ΣX, ΣY, and ΣZ.

Hence then, if a body of finite magnitude be acted on by any forces, its center of gravity will move as a particle of equal mass would if acted on by all the forces impressed on the body.

We also see that the motion of the center of gravity of the body will not be affected by changing the points of application of any of the impressed forces, provided their intensities and directions are not altered.

If all the forces X, Y and Z be given explicitly as functions of the variables, these equations will give the motion of the center of gravity at once by integration.

When this is not the case, that is, when unknown forces enter, arising from geometrical conditions to which the motion is subject, it will generally be necessary to combine them with the remaining three equations before the integration can be effected.

93. Substituting $\bar{x}+x'$, $\bar{y}+y'$, $\bar{z}+z'$ for x, y, and z in equations (II) we have

$$\Sigma\left\{(\bar{y}+y')\left(Z-m\frac{d^2\bar{z}}{dt^2}-m\frac{d^2z'}{dt^2}\right)-(\bar{z}+z')\left(Y-m\frac{d^2\bar{y}}{dt^2}-m\frac{d^2y'}{dt^2}\right)\right\}=0,$$

$$\Sigma\left\{(\bar{z}+z')\left(X-m\frac{d^2\bar{x}}{dt^2}-m\frac{d^2x'}{dt^2}\right)-(\bar{x}+x')\left(Z-m\frac{d^2\bar{z}}{dt^2}-m\frac{d^2z'}{dt^2}\right)\right\}=0,$$

$$\Sigma\left\{(\bar{x}+x')\left(Y-m\frac{d^2\bar{y}}{dt^2}-m\frac{d^2y'}{dt^2}\right)-(\bar{y}+y')\left(X-m\frac{d^2\bar{x}}{dt^2}-m\frac{d^2x'}{dt^2}\right)\right\}=0.$$

Now in these equations $\bar{x}$, $\bar{y}$, $\bar{z}$ and their differential coefficients are common to all the particles of the system, they may therefore be placed before the symbol Σ.

Also, $\Sigma mx'=0$, $\Sigma my'=0$, $\Sigma mz'=0$;

and therefore,

$$\Sigma m\frac{d^2x'}{dt^2}=0, \quad \Sigma m\frac{d^2y'}{dt^2}=0, \quad \Sigma m\frac{d^2z'}{dt^2}=0.$$

By means of these relations, and also the equations (III), the preceding equations may be reduced to

$$\left.\begin{aligned}\Sigma\left\{y'\left(Z-m\frac{d^2z'}{dt^2}\right)-z'\left(Y-m\frac{d^2y'}{dt^2}\right)\right\}&=0,\\ \Sigma\left\{z'\left(X-m\frac{d^2x'}{dt^2}\right)-x'\left(Z-m\frac{d^2z'}{dt^2}\right)\right\}&=0,\\ \Sigma\left\{x'\left(Y-m\frac{d^2y'}{dt^2}\right)-y'\left(X-m\frac{d^2x'}{dt^2}\right)\right\}&=0.\end{aligned}\right\}\quad\ldots(\text{IV})$$

Now we observe that the position of the center of gravity does not enter into these equations either explicitly or implicitly.

If then the center of gravity of the body were fixed, and the body were acted on at the same points by the same impressed forces that act on it when it is free, the equations of motion would be the same, since the forces introduced by the condition of the center of gravity being at rest would all act at the center of gravity, the co-ordinates of their points of application would be zero, and they would therefore not enter into the equations.

Hence then we conclude that the motion of the body about its center of gravity is the same as if the center of gravity were fixed, and the body acted on by the same impressed forces.

These two results are expressed by saying that motions of translation and rotation are independent. They are of the utmost importance in the solution of every problem about the motion of one rigid body.

We will for distinctness enunciate them again.

When a body is in motion under the action of any forces, its center of gravity moves as if the whole mass were collected there and acted on by all the forces which act on the body. And the motion of the body about its center of gravity is the same as if the center of gravity were fixed, and the body acted on by the same forces that do act on it.

94. The equations (IV) involve the co-ordinates of every particle in the body. To determine from them the motion of the body, they must be reduced to others containing only co-ordinates common to the whole body.

We shall not enter upon this reduction in the general case at present, as the process is complicated, but shall confine our attention to the cases where all the particles of the body move in parallel planes.

Let the plane of xy be that to which the motion is parallel; then the ordinate parallel to z of every particle is constant. Equations (IV) are therefore reduced to

$$\Sigma\left\{x'\left(Y-m\frac{d^2y'}{dt^2}\right)-y'\left(X-m\frac{d^2x'}{dt^2}\right)\right\}=0,$$

$$\text{or, } \Sigma m\left(x'\frac{d^2y'}{dt^2}-y'\frac{d^2x'}{dt^2}\right)=\Sigma\left(x'Y-y'X\right).$$

Let $$x'=r\cos\theta, \quad y'=r\sin\theta,$$

then for each particle r is constant; so that we have

$$\frac{d^2x'}{dt^2}=-r\sin\theta\frac{d^2\theta}{dt^2}-r\cos\theta\left(\frac{d\theta}{dt}\right)^2,$$

$$\frac{d^2y'}{dt^2}=\quad r\cos\theta\frac{d^2\theta}{dt^2}-r\sin\theta\left(\frac{d\theta}{dt}\right)^2.$$

Making these substitutions, we have

$$\Sigma mr^2\frac{d^2\theta}{dt^2}=\Sigma\left(x'Y-y'X\right).$$

Now since it is a rigid body $\frac{d\theta}{dt}$, and therefore $\frac{d^2\theta}{dt^2}$ is the same for every particle of the body, we may therefore place it before Σ. So that we have

$$\frac{d^2\theta}{dt^2}\ \Sigma mr^2=\Sigma\left(x'Y-y'X\right).$$

Now the right-hand side of this equation is the resultant moment of the impressed forces round an axis through the center of gravity

perpendicular to the plane of motion; calling this L and putting Mk^2 for Σmr^2, we have

$$Mk^2\frac{d^2\theta}{dt^2}=L.$$

Combining with this the first two of equations (III), we have

$$\left.\begin{aligned} M\frac{d^2\bar{x}}{dt^2}&=\Sigma X,\\ M\frac{d^2\bar{y}}{dt^2}&=\Sigma Y,\\ Mk^2\frac{d^2\theta}{dt^2}&=L. \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(V)}$$

These three equations are sufficient to determine the motion of a rigid body which takes place in one plane, whenever the forces which act on it are given as functions of the variables.

This is seldom the case. Unknown forces generally enter into these equations, arising from conditions to which the motion of the body is subject. There will, however, in every case be a geometrical equation corresponding to each unknown force so introduced, so that the number of equations will always be equal to the number of unknown quantities.

In effecting this last transformation all the geometrical conditions of the rigidity of the body were introduced when we considered r to be constant for each particle, and $\frac{d\theta}{dt}$ to be the same for all the particles.

95. In the preceding reduction the expression $\frac{d^2\theta}{dt^2}\Sigma mr^2$ occurs, where r is the distance of the particle m from a certain fixed line in the body. mr^2 is called the "Moment of Inertia" of the particle m round this line, and Σmr^2 is the moment of inertia of the whole body round the same line; it is the sum of the moments of inertia of every particle. This quantity will frequently occur; it is different for different bodies and for different lines in the same body. It is generally written Mk^2, in which case k is called the "Radius of Gyration" of the body round the particular line. The Moment of Inertia of any body may be found by the ordinary methods of integration. Again, $\frac{d\theta}{dt}$ is called the angular velocity of the body; if we conceive a plane fixed in the body perpendicular to the plane of motion to be inclined at an angle θ to a fixed plane also perpen-

dicular to the plane of motion, $\frac{d\theta}{dt}$ is the rate at which the angle θ is increasing, and the propriety of the term angular velocity is manifest. Sometimes $\frac{d^2\theta}{dt^2}$ is called the angular accelerating force.

96. Let us now take an example of the application of these equations in the case of motion in one plane.

A uniform heavy rod slides down between a smooth vertical and a smooth horizontal plane, to determine the motion.

Let the figure represent the rod at the time t.

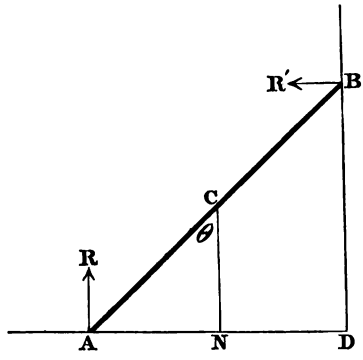
$AB = 2a$,

$DN = x$, $NC = y$,

$ACN = \theta$,

M the mass of the rod,

R, R' the reactions of the planes, then from equations (V) we have



$$M\frac{d^2x}{dt^2} = R', \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$M\frac{d^2y}{dt^2} = R - Mg, \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$Mk^2\frac{d^2\theta}{dt^2} = Ra\sin\theta - R'a\cos\theta; \quad \ldots\ldots (3)$$

and corresponding to the two unknown forces R, R' we have the two geometrical relations

$$x = a\sin\theta, \quad y = a\cos\theta, \quad \ldots\ldots\ldots (4)$$

which express that the ends of the rod are in contact with the planes.

Eliminating R and R' from the equations, we get

$$2M\left(\frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{d^2y}{dt^2}\frac{dy}{dt} + k^2\frac{d^2\theta}{dt^2}\frac{d\theta}{dt}\right) = -2Mg\frac{dy}{dt}.$$

Integrating this, we have

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + k^2\left(\frac{d\theta}{dt}\right)^2 = C - 2gy,$$

and using equations (4),

$$(a^2 + k^2)\left(\frac{d\theta}{dt}\right)^2 = C - 2ag\cos\theta.$$

If we suppose the sliding to commence when the rod is vertical, we have $\frac{d\theta}{dt}=0$ when $\theta=0$;

$$\therefore\ 0=C-2ag;$$

$$\therefore \left(\frac{d\theta}{dt}\right)^2=\frac{2ag}{a^2+k^2}(1-\cos\theta);$$

$$\therefore \frac{d\theta}{dt}=\sqrt{\frac{4ag}{a^2+k^2}}\sin\frac{\theta}{2}\ldots\ldots\ldots(5)$$

Or, since $k^2=\frac{a^2}{3}$,

$$\frac{d\theta}{dt}=\sqrt{\frac{3g}{a}}\sin\frac{\theta}{2},$$

which gives the angular velocity in any position; the equations

$$\frac{dx}{dt}=a\cos\theta\frac{d\theta}{dt},\quad \frac{dy}{dt}=-a\sin\theta\frac{d\theta}{dt},$$

give the velocity of the center of gravity.

The integral of equation (5) is

$$\log\tan\frac{\theta}{4}=\sqrt{\frac{3g}{4a}}t+C.$$

On attempting to determine C by the condition used before, we get $C=-\infty$. This might have been expected since the position from which the motion was assumed to commence is one of equilibrium, so that the motion would not take place at all. We may, however, suppose the motion to commence in a position differing very slightly from that of equilibrium, in which case the velocity we have obtained will only be affected by a small quantity of the same order of magnitude, and C will be rendered finite: or we may suppose a velocity equal to that indicated in the solution to have been communicated to the rod in the corresponding position.

To find the pressures on the two planes in any position, we have

$$\left(\frac{d\theta}{dt}\right)^2=\frac{3g}{2a}(1-\cos\theta);$$

$$\therefore \frac{d^2\theta}{dt^2}=\frac{3g}{4a}\sin\theta,$$

also from (4) $\quad x=a\sin\theta,\ y=a\cos\theta;$

$$\therefore \frac{d^2x}{dt^2}=\frac{3g}{4}\sin\theta(3\cos\theta-2),$$

$$\frac{d^2y}{dt^2}=-\frac{3g}{4}(1+2\cos\theta-3\cos^2\theta),$$

substituting these values in equations (1) and (2), we have

$$R' = \frac{3Mg}{4} \sin\theta (3\cos\theta - 2),$$

$$R = \frac{Mg}{4} (1 - 6\cos\theta + 9\cos^2\theta),$$

$$= \frac{Mg}{4} \{1 + 3\cos\theta (3\cos\theta - 2)\},$$

which give the pressures on the planes corresponding to any value of θ.

When $3\cos\theta - 2 = 0$, $R' = 0$, and as θ increases R' becomes negative: this result shews that a force would be required to keep the point B in contact with the vertical plane: as this force is not supposed to exist the rod will at that point leave the plane, and the motion will be different afterwards.

From the expression for R we see that it continues positive up to this point.

To determine the motion after the rod leaves the plane BD we have the equations

$$M\frac{d^2x}{dt^2} = 0,$$

$$M\frac{d^2y}{dt^2} = R - Mg,$$

$$Mk^2\frac{d^2\theta}{dt^2} = Ra\sin\theta,$$

$$y = a\cos\theta.$$

From the first equation it appears that $\frac{dx}{dt}$ remains constant with the value it had when the rod left the plane.

Now at that time

$$\frac{dx}{dt} = a\cos\theta\frac{d\theta}{dt},$$

$$= \cos\theta\sqrt{\frac{3ag}{2}(1-\cos\theta)},$$

and putting for $\cos\theta$ its value $\frac{2}{3}$ we get

$$\frac{dx}{dt} = \tfrac{1}{3}\sqrt{2ag}.$$

This will continue to be the velocity of the center of gravity in a horizontal direction.

From the other equations we have

$$\left(\frac{dy}{dt}\right)^2 + k^2\left(\frac{d\theta}{dt}\right)^2 = C - 2ag\cos\theta;$$

$$\therefore\ a^2(\sin^2\theta + \tfrac{1}{3})\left(\frac{d\theta}{dt}\right)^2 = C - 2ag\cos\theta.$$

Now when the rod left the vertical plane

$$\cos\theta = \tfrac{2}{3},\ \left(\frac{d\theta}{dt}\right)^2 = \frac{g}{2a};$$

$$\therefore \frac{4ag}{9} = C - \frac{4ag}{3};$$

$$\therefore\ C = \frac{16ag}{9};$$

$$\therefore\ (3\sin^2\theta + 1)\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{3a}(8 - 9\cos\theta),$$

which gives the value of $\frac{d\theta}{dt}$ in any position, and we may find the value of R as in the previous case.

97. When the motion of every particle is parallel to one plane, we have only the three equations (V) instead of the six equations (III) and (IV).

These six equations however must always be satisfied, let us therefore consider what they become in this case.

The last of equations (III) becomes

$$\Sigma Z = 0,$$

which shews that the sum of the resolved parts parallel to the axis of z of all the forces which act on the body must be zero.

If the body is free this is one of the conditions that must hold among the forces acting on it that the motion may be of the kind we have supposed.

If the body is constrained to move in the manner supposed this will be one of the equations which determine the forces called into action by this constraint.

The first two of equations (IV) become

$$\Sigma m z' \frac{d^2y'}{dt^2} = \Sigma(z'Y - y'Z),$$

$$\Sigma m z' \frac{d^2x'}{dt^2} = \Sigma(z'X - x'Z).$$

Now from the relations

$$x' = r\cos\theta,\ \ y' = r\sin\theta,$$

the left-hand sides of these equations can be expressed in terms of $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$, which are known by (V); and the equations then give two more conditions to be satisfied by the impressed forces.

Thus whenever the motion is of this sort, the six equations (III) and (IV) are reduced by relations between the impressed forces to no more than three independent equations.

98. It frequently happens that the conditions of equilibrium of the system under consideration have been found in a form simpler than the six general equations of equilibrium. Now, as in considering any particular case of equilibrium we may employ either the six general equations, or the more simple ones that have been deduced for the class to which the particular case under consideration belongs, so we may employ either set of equations of equilibrium to form the equations of motion.

The case of a body moving about a fixed axis affords an illustration of this remark, and deserves consideration from its own importance.

In the case of a body moveable about a fixed axis, we may determine the conditions of equilibrium from the six general equations, in which case we introduce unknown forces for the pressures exerted by the axis on the body; or we may take the condition which has been deduced from these, that the body will be in equilibrium when the resultant moment round the fixed axis is zero.

We will consider the motion of a body about a fixed axis in both these points of view.

Let r be the distance from the axis of a particle whose mass is m, since its motion is at every instant perpendicular to the axis and the radius vector, the resolved part of the effective force in this direction is

$$m\frac{d^2s}{dt^2} \text{ which } = mr\frac{d^2\theta}{dt^2}.$$

If therefore we apply a force equal to this in the opposite direction, its moment round the axis will be $-mr^2\frac{d^2\theta}{dt^2}$.

Let L be the whole moment of the impressed forces, then the condition that the whole moment round the axis is zero gives the equation

$$L - \Sigma mr^2\frac{d^2\theta}{dt^2} = 0,$$

or, since $\frac{d\theta}{dt}$, and therefore $\frac{d^2\theta}{dt^2}$ is common to every particle of the system

$$L - \frac{d^2\theta}{dt^2}\Sigma mr^2 = 0,$$

or, putting Mk^2 for Σmr^2

$$Mk^2\frac{d^2\theta}{dt^2} = L.$$

From this equation the motion of the body may be completely determined.

We will now see how the same result may be obtained from the six general equations.

Let the fixed axis be taken as the axis of z, and a plane perpendicular to it through the center of gravity for the plane of xy; let $\bar{x}, \bar{y}$ be the co-ordinates of the center of gravity, θ the angle which a line joining the center of gravity and the origin makes with the axis of x.

We may suppose the body to be fixed to the axis at two points, let these be at distances c, and c' from the origin, and let the resolved parts parallel to the axes of the forces which these points exercise on the body be R_x, R_y, R_z, R_x', R_y', R_z'.

Then we have from equations (III), since $\frac{d^2\bar{z}}{dt^2} = 0$,

$$M\frac{d^2\bar{x}}{dt^2} = \Sigma X + R_x + R_x' \ldots\ldots (1)$$

$$M\frac{d^2\bar{y}}{dt^2} = \Sigma Y + R_y + R_y' \ldots\ldots (2)$$

$$0 = \Sigma Z + R_z + R_z' \ldots\ldots (3)$$

also from equations (II), since $\frac{d^2z}{dt^2} = 0$ for every particle, we have

$$-cR_y - c'R_y' + \Sigma(yZ - zY) + \Sigma mz\frac{d^2y}{dt^2} = 0 \ldots\ldots (4)$$

$$cR_x + c'R_x' + \Sigma(zX - xZ) - \Sigma mz\frac{d^2x}{dt^2} = 0 \ldots\ldots (5)$$

$$\Sigma(xY - yX) - \Sigma\left(mx\frac{d^2y}{dt^2} - my\frac{d^2x}{dt^2}\right) = 0 \ldots\ldots (6)$$

equation (6) may be reduced to

$$Mk^2\frac{d^2\theta}{dt^2} = \Sigma(xY - yX).$$

This is the same as the equation

$$Mk^2 \frac{d^2\theta}{dt^2} = L,$$

which was obtained by the other method: it is sufficient to determine the motion.

The remaining five equations enable us to determine the unknown forces R_x, R_y, R_x', R_y', and $R_z + R_z'$; just as in the statical problem the general method gives the pressures on the axis, while the particular method tacitly eliminates them.

We will return to these equations, and put them into a more convenient form for determining the unknown pressures in particular cases, but for the present will consider the motion of the body.

99. A particular case of motion round a fixed axis worthy of consideration is where the only impressed force is gravity, and the axis is horizontal.

Let h be the distance of the center of gravity from the fixed axis, θ the angle between a vertical plane through the axis, and a plane through the axis and the center of gravity.

Then $L = -Mgh \sin\theta$, so that the equation is

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{k^2} \sin\theta; \ldots\ldots\ldots\ldots (1)$$

$$\therefore \left(\frac{d\theta}{dt}\right)^2 = C + 2\frac{gh}{k^2} \cos\theta \ldots\ldots\ldots (2).$$

C must be determined from the angular velocity corresponding to some known value of θ.

Now the equation of motion of a point in a circular arc of radius l is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin\theta;$$

comparing this with equation (1) we see that they are identical, if

$$l = \frac{k^2}{h}.$$

Hence then the time of oscillation of the rigid body will be the same as that of a particle attached to a string whose length is $\frac{k^2}{h}$, for the same amplitude. This is called the isochronous simple pendulum, the body itself being called a compound pendulum.

The equation $\frac{d^2\theta}{dt^2} + \frac{gh}{k^2} \sin\theta = 0$

can only be solved so as to give the relation between θ and t in a series.

When however θ is always small, so that powers higher than the first may be neglected, we can find an approximate solution. Writing θ for $\sin\theta$, which amounts to neglecting θ^3, we have

$$\frac{d^2\theta}{dt^2} + \frac{gh\theta}{k^2} = 0 \ldots\ldots\ldots\ldots\ldots (3);$$

$$\therefore\ \theta = A\cos\left(\sqrt{\frac{gh}{k^2}}\, t + B\right);$$

$$\text{and } \frac{d\theta}{dt} = -A\sqrt{\frac{gh}{k^2}} \sin\left(\sqrt{\frac{gh}{k^2}}\, t + B\right).$$

Let the body start from rest when $\theta = \alpha$, then

$$0 = -A\sqrt{\frac{gh}{k^2}} \sin B,$$

$$\alpha = A\cos B;$$

$$\therefore\ B = 0 \text{ and } A = \alpha;$$

$$\therefore\ \theta = \alpha\cos\sqrt{\frac{gh}{k^2}}\, t;$$

when $\theta = 0$ we must have

$$\sqrt{\frac{gh}{k^2}}\, t = (2n+1)\frac{\pi}{2};$$

let t_1, t_2, &c. be the successive corresponding values of t, then we have

$$t_1 = \frac{\pi}{2}\sqrt{\frac{k^2}{gh}},$$

$$t_2 = \frac{3\pi}{2}\sqrt{\frac{k^2}{gh}},$$

$$t_3 = \frac{5\pi}{2}\sqrt{\frac{k^2}{gh}},$$

$$\ldots = \ldots\ldots$$

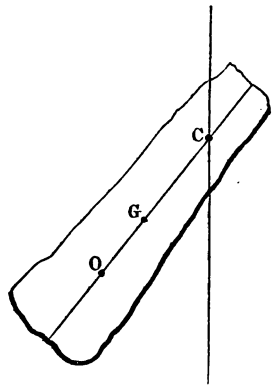
$$\therefore\ t_2 - t_1 = t_3 - t_2 = \&c. = \pi\sqrt{\frac{k^2}{gh}}.$$

This shews that the center of gravity of the body will attain the lowest position after successive intervals of $\pi\sqrt{\frac{k^2}{gh}}$.

We also see that, to the order of approximation to which we have proceeded, the time is independent of the value of a, or the amplitude of the oscillation.

We shall return to this subject afterwards when we treat of pendula, and shall then shew how a more approximate value of the time of oscillation may be obtained; before leaving it, however, there are a few general properties which deserve notice.

Let the figure represent a section of the body made by a plane through its center of gravity perpendicular to the axis round which it oscillates; and let C be the point in which the axis cuts this plane, G the center of gravity.



The two equations

$$\frac{d^2\theta}{dt^2} + \frac{gh}{k^2}\sin\theta = 0;$$

and $$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0,$$

will be identical if $l = \frac{k^2}{h}$.

If therefore we take in CG produced a point O such that $CO = l$, the position and motion at any time will be the same under the same initial circumstances as if the whole mass were collected at O.

O is called the center of oscillation.

If k_1 be the radius of gyration round an axis through G parallel to the fixed one through C,

$$k^2 = k_1^2 + h^2,$$

and therefore $$l = \frac{k_1^2}{h} + h.$$

Now if the body were suspended by an axis through O parallel to the same axis, the length of the corresponding isochronous simple pendulum would be

$$\frac{k_1^2}{l-h} + l - h,$$

which is equal to l.

Hence then oscillations about this axis take place in the same time as those of the same amplitude about the first axis through C.

C is called the center of suspension, and this property is expressed by saying that the centers of oscillation and suspension are reciprocal.

This property might also have been deduced in the following manner. Suppose it required to find the distance from G of an axis parallel to a given direction, about which the oscillations will take place in the same time as those of a simple pendulum whose length is l.

Let h be this distance; then h will be determined from the equation

$$l = \frac{k_1^2}{h} + h,$$

$$\text{or } h^2 - lh + k_1^2 = 0.$$

Now this being a quadratic equation gives two values of h, h_1 and h_2 such that $h_1 + h_2 = l$.

So that an oscillation about an axis at distance h_1 takes place in the same time as one about an axis at distance $l - h_1$.

Also, since in this investigation nothing has been introduced to indicate in which direction from G the required axis lies, we conclude that any direction will satisfy the condition. If therefore with center G we describe circles in the plane of the paper at distances GO and GC, small oscillations will take place in the same time about any parallel axis through any point in either of these circles.

The value of h which will make l, and therefore the time of oscillation a minimum may be easily determined: since

$$l = \frac{k_1^2}{h} + h,$$

that l may be a minimum, we have

$$-\frac{k_1^2}{h^2} + 1 = 0,$$

$$\text{or } h = k_1,$$

$$\text{and therefore } l = 2h \text{ or } 2k_1.$$

In this case therefore the two circles we have mentioned coincide.

The axes determined by these conditions are not axes about which the time of oscillation is the least possible for the body, but only for axes in the body parallel to a given direction. To find those about which it will be absolutely the least, we must select that direction for which k is the least. This we know will be one of the principal axes through the center of gravity of the body.

CHAPTER VII.

100. To find the conditions of equilibrium of a system of rigid bodies we have for each body of the system six equations of equilibrium, as if that particular body were the only one of the system. And besides these we have equations expressing the geometrical connexions of the bodies.

The same method must be adopted to determine the motion of a system of rigid bodies; we must write down for each body of the system the six equations (III) and (IV), or the three equations (V), and then the equations expressing the geometrical conditions which the bodies satisfy.

When the equations are once written down the difficulties of solving them are merely analytical, and such artifices must be used as the forms of the equations suggest.

The constants introduced in integration must be determined in the same way as in the case of a single body.

When any system is in equilibrium we may either consider each body separately, and write down for it the six equations of equilibrium; in which case the actions of the other bodies of the system on it will enter as external forces; or we may consider the whole system as one body and write down for it the six equations of equilibrium; in which case the mutual actions of the different parts of the system will not appear in the equations; or we may consider any two or more of the bodies as one, and write down for them the equations of equilibrium.

These different methods may also be adopted in Dynamics. The dynamical equations, as first written down, are strictly equations of equilibrium, and it is only from the analytical character of the quantities that enter them that they enable us to determine the motion of the system.

Though the methods to be employed in solving the equations of motion must in each case be suggested by the forms of those equations, there are certain artifices by which the first integrals may often be obtained, the applicability of which can be determined at once from an inspection of the system. The difficulty generally arises from there being unknown forces which have to be eliminated

by means of geometrical relations. These artifices enable us in certain cases to effect that elimination at once.

101. When a body or system of bodies is in motion under the action of any forces, if $\bar{x}, \bar{y}, \bar{z}$ are the co-ordinates of the center of gravity at the time t, and M the mass of the system, we have the equations

$$\left.\begin{aligned} M\frac{d^2\bar{x}}{dt^2} &= \Sigma X, \\ M\frac{d^2\bar{y}}{dt^2} &= \Sigma Y, \\ M\frac{d^2\bar{z}}{dt^2} &= \Sigma Z; \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(III)}$$

in which no forces appear but such as arise from causes external to the system; that is, no pressure of one part of the system against another enters, no mutual attraction, no tension of a string connecting two parts of the system. But all forces do enter which arise from causes external to the system, as pressures of fixed surfaces, or fixed points, tensions of strings passing over fixed pulleys, &c., together with the given external forces whatever they may be.

Now equations (III) shew that when

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

the center of gravity will move uniformly in a straight line. Thus, whatever the internal forces of the system, so long as there are no external forces the center of gravity will move in a straight line with uniform velocity. This is called the Principle of the Conservation of the motion of the center of gravity.

We see from equations (III) that if $\Sigma X = 0$, $\frac{d\bar{x}}{dt}$ is constant, whatever ΣY and ΣZ are. Hence if the resolved parts of all the external forces acting on a system parallel to any line have a resultant zero, the motion of the center of gravity parallel to that line will be uniform. Similarly, if two of them vanish the resolved part of the motion parallel to that plane will be uniform and rectilinear. Thus, for instance, if any number of balls be piled up on a perfectly smooth horizontal plane and then left to themselves, they will be acted on by no external horizontal force, they will therefore fall down in such a manner that their center of gravity will describe a vertical line.

In the cases that will most frequently occur the center of gravity of the system will be at rest at the commencement of the motion, and will therefore continue so throughout.

102. The next principle that we shall mention depends on equations (II), which may be written

$$\left.\begin{aligned} \Sigma m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)&=\Sigma(yZ-zY),\\ \Sigma m\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right)&=\Sigma(zX-xZ),\\ \Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)&=\Sigma(xY-yX). \end{aligned}\right\}\ \ldots\ (\mathrm{II})$$

Whenever the right-hand sides of these equations are zero the equations themselves become

$$\left.\begin{aligned} \Sigma m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)&=0,\\ \Sigma m\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right)&=0,\\ \Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)&=0, \end{aligned}\right\}\ \ldots\ (1)$$

the integrals of which are

$$\left.\begin{aligned} \Sigma m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)&=h_1,\\ \Sigma m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right)&=h_2,\\ \Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)&=h_3. \end{aligned}\right\}\ \ldots\ (2)$$

Now if A_x be the area described in any time by the projection on the plane of yz of the line joining the origin and the particle whose co-ordinates are x, y, z,

$$y\frac{dz}{dt}-z\frac{dy}{dt}=2\frac{dA_x}{dt}.$$

Employing a similar notation for the other co-ordinate planes, and substituting in equations (2), we have

$$\left.\begin{aligned} 2\Sigma m\frac{dA_x}{dt}&=h_1,\\ 2\Sigma m\frac{dA_y}{dt}&=h_2,\\ 2\Sigma m\frac{dA_z}{dt}&=h_3, \end{aligned}\right\}\ \ldots\ (3)$$

from which
$$\begin{aligned} 2\Sigma mA_x&=h_1t+C_1,\\ 2\Sigma mA_y&=h_2t+C_2,\\ 2\Sigma mA_z&=h_3t+C_3, \end{aligned}$$

or, if we suppose A_x, &c., to be the areas described since the instant from which t is measured,

$$\left.\begin{aligned} 2\Sigma m A_x &= h_1 t, \\ 2\Sigma m A_y &= h_2 t, \\ 2\Sigma m A_z &= h_3 t. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (4)$$

In these equations or in equations (3), consists the principle which is called the Principle of the Conservation of Areas.

We may state it in words as follows:

The area described by the projection on the plane of yz of the radius vector of any particle, is called the area described by the particle round the axis of x. Since any fixed line may be taken as the axis of x, this definition is applicable to any line. In certain cases the sum of the products of the mass of each particle of the system, and the area described by that particle round some fixed line is proportional to the time; whenever this is the case, the principle of the conservation of areas is said to hold for that system, round that line.

This may be the case for only one line, or for any line whatever, or for lines satisfying certain conditions; let us consider generally how to determine in what cases it does hold.

We see by the equations from which it is deduced, that it holds for the axis of x whenever

$$\Sigma (yZ - zY) = 0.$$

Now in this expression no mutual actions or other internal forces enter. It is the expression for the resultant moment of the impressed forces round the axis of x. The principle will therefore hold for any line round which the moment of the impressed forces is zero.

First, let there be no external forces, or such as would keep the system in equilibrium in any position, if placed at rest in that position. In this case whatever line be chosen, the moment of the forces round that line is zero, and consequently the principle holds for any line whatever.

Again, let the resultant of the impressed forces always pass through a fixed point. In this case the moment round any line through that point is zero, and therefore the principle holds for any line through that point.

If all the impressed forces which act on the system are parallel, the principle will hold round any line parallel to their common direction.

If the resultant of the impressed forces always passes through a fixed line, the principle will hold round that line. It is needless to enumerate any more particular cases.

103. When the resultant of the impressed forces acting on the system is either zero, or passes always through a fixed point, we have

$$2\Sigma m A_x = h_1 t,$$
$$2\Sigma m A_y = h_2 t,$$
$$2\Sigma m A_z = h_3 t,$$

for any directions whatever of the axes; and in the former case for any origin, in the latter only when the fixed point is origin.

h_1, h_2, h_3, are constants introduced by integration, which may be determined from the circumstances of the motion at some known instant; and will be different for different axes of co-ordinates.

If λ, μ, ν be the direction-cosines of any line passing through the origin, the sum of the products of the masses and areas described by them round this line

$$= 2\Sigma m(\lambda A_x + \mu A_y + \nu A_z)$$
$$= \lambda \cdot 2\Sigma m A_x + \mu \cdot 2\Sigma m A_y + \nu \cdot 2\Sigma m A_z$$
$$= (\lambda h_1 + \mu h_2 + \nu h_3) t$$
$$= \left(\lambda \frac{h_1}{h} + \mu \frac{h_2}{h} + \nu \frac{h_3}{h}\right) h t;$$

where $h^2 = h_1^2 + h_2^2 + h_3^2$.

Now $\frac{h_1}{h}$, $\frac{h_2}{h}$, $\frac{h_3}{h}$ may be considered as the direction-cosines of some line, and if θ be the angle between this line and that round which the areas are reckoned, the sum in question is $ht\cos\theta$.

The greatest value of this is when $\cos\theta = 1$, which is the case when

$$\lambda = \frac{h_1}{h}, \quad \mu = \frac{h_2}{h}, \quad \nu = \frac{h_3}{h}.$$

Hence then the line whose direction-cosines are $\frac{h_1}{h}$, $\frac{h_2}{h}$, $\frac{h_3}{h}$ is such that the sum of the products of the areas round it and the masses is a maximum.

Since its direction-cosines are independent of t its position is constant, and from its nature its direction in space is independent of the axes to which it is referred. We see therefore that in every such system there exists a line whose direction in space is invariable

and can be determined at any instant, from the circumstances of the system at that instant.

A plane perpendicular to this line is called the "invariable plane," and the line itself is called the "invariable axis."

These results agree with those which were determined previously in a particular case.

The sum of these products round any line inclined at an angle θ to this is $ht\cos\theta$; and therefore becomes zero for any of the lines at right angles to this one.

Many other curious properties of the motion connected with this line might be deduced from these equations, these, however, are the most important.

104. When a system is in equilibrium under the action of forces X, Y, Z, &c. We have by the principle of virtual velocities

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = 0.$$

We have therefore by D'Alembert's principle for a system in motion

$$\Sigma\left\{\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right\} = 0.$$

We will make a few remarks on this equation.

No internal forces appear in it which arise from invariable geometrical connexions of the parts, that is, which arise from connexions of the parts which are geometrically the same before and after the indefinitely small displacement which is supposed to take place. Such, for instance, as the tensions of flexible and inextensible strings; the pressures of smooth surfaces which continue in contact, or of rough surfaces where the displacement may have been supposed to take place by rolling; the pressures of fixed points, &c. Also we may omit any external forces, the displacements of the points of application of which are such that their virtual velocities are zero.

Now since in this equation the displacement and therefore the values of δx, δy, &c. are arbitrary, being subject only to the condition of not violating the geometrical relations referred to above, we may suppose the displacement to be that which actually takes place in the indefinitely small time δt, that is, for $\delta x, \delta y, \delta z \ldots\ldots$

we may write $\frac{dx}{dt}\delta t$, $\frac{dy}{dt}\delta t$, $\frac{dz}{dt}\delta t, \ldots\ldots$

provided the geometrical relations of the system at the beginning and end of the time δt are the same. The equation thus becomes

$$\Sigma m\left(\frac{d^2x}{dt^2}\frac{dx}{dt}+\frac{d^2y}{dt^2}\frac{dy}{dt}+\frac{d^2z}{dt^2}\frac{dz}{dt}\right)\delta t=\Sigma\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right)\delta t,$$

the integral of which is

$$\Sigma m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}=C+2\int\Sigma\left(Xdx+Ydy+Zdz\right),$$

$$\text{or } \Sigma mv^2=C+2\int\Sigma\left(Xdx+Ydy+Zdz\right).$$

Now in this equation no forces appear which are either internal forces such as we have already mentioned, or external forces such that their virtual velocities arising from the actual displacement are zero. Such forces are the tensions of inextensible strings attached to fixed points; pressures of smooth fixed surfaces; pressures of fixed points against smooth surfaces, &c., where the actual displacement of the point of application of the force is perpendicular to the direction of the force; and also forces where there is no displacement of the point of application of the force, as when a body moves about a fixed point or axis; or rolls without sliding on a rough surface, &c.

We may remark that the internal forces do not enter into the equation, because if they did for each force there would be an equal force whose virtual velocity would have a contrary sign, so that they would enter in pairs which would destroy each other; and that the external forces which we have mentioned do not enter because the virtual velocity of each is zero.

The product of the mass of a particle and the square of the absolute velocity with which it is moving is called the *vis viva* of the particle.

Hence, then, Σmv^2 is the *vis viva* of the whole system, and

$$\Sigma mv^2=C+2\int\Sigma\left(Xdx+Ydy+Zdz\right)$$

is an equation which gives the *vis viva* of the whole system in any position, whenever the integration on the right-hand side can be performed. This is easily done in most of the cases that will occur in practice.

In order that the integration may be possible, the expression

$$\Sigma\left(Xdx+Ydy+Zdz\right)$$

must be a perfect differential of some function of the co-ordinates of the different particles.

This will be the case whenever the forces X, Y, Z, &c. are constant, or arise from the attractions of fixed centers, or from the

mutual attractions of the different particles of the system, with forces varying as some powers of the distance. It will not be the case when the forces arise from the attractions of moveable centers not forming part of the system, or when the forces are functions of the velocities, as when the motion takes place in a resisting medium, &c.

Let it be a perfect differential, and let its integral be

$$\phi(x_1, y_1, z_1, x_2, y_2, z_2 \ldots),$$

$$\text{then } \Sigma m v^2 = C + 2\phi(x_1, y_1, z_1 \ldots)$$

Let $a_1, b_1, c_1 \ldots$ be the values of $x_1, y_1, z_1 \ldots$ when the velocities of the different particles are $V_1, V_2 \ldots$ then

$$\Sigma m v^2 - \Sigma m V^2 = 2\phi(x_1, y_1 \ldots) - 2\phi(a_1, b_1 \ldots)$$

This equation shews that the change in the *vis viva* of the system which takes place in any time depends only on the co-ordinates of the particles at the beginning and end of that time, and not on their velocities or directions of motion.

This is called the Principle of *vis viva*.

If there are no forces external to the system but such as do not enter into the equation of virtual velocites, that is, if

$$\Sigma(X dx + Y dy + Z dz) = 0, \quad \Sigma m v^2 = C,$$

or the *vis viva* of the system is constant. This is called the Principle of the Conservation of *vis viva*.

It is convenient to express the *vis viva* of the system in terms of that due to the motion of translation of the center of gravity, and that due to the motion of rotation about it.

Preserving the notation already used, and writing $\bar{x} + x'$ for x, &c., we have

$$\frac{dx}{dt} = \frac{d\bar{x}}{dt} + \frac{dx'}{dt},$$

$$\frac{dy}{dt} = \frac{d\bar{y}}{dt} + \frac{dy'}{dt},$$

$$\frac{dz}{dt} = \frac{d\bar{z}}{dt} + \frac{dz'}{dt};$$

and substituting these, we get

$$\Sigma m \left\{ \left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2 + \left(\frac{d\bar{z}}{dt}\right)^2 + \left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + \left(\frac{dz'}{dt}\right)^2 + 2\frac{d\bar{x}}{dt}\frac{dx'}{dt} + 2\frac{d\bar{y}}{dt}\frac{dy'}{dt} + 2\frac{d\bar{z}}{dt}\frac{dz'}{dt} \right\},$$

as the expression for the *vis viva*; or bearing in mind that $\bar{x}, \bar{y}, \bar{z}$ are the same for every particle of the system, and also that

$$\Sigma m x' = 0, \ \Sigma m y' = 0, \ \Sigma m z' = 0,$$

and therefore $\Sigma m \frac{dx'}{dt} = 0, \ \Sigma m \frac{dy'}{dt} = 0, \ \Sigma m \frac{dz'}{dt} = 0,$

we have

$$M\left\{\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2 + \left(\frac{d\bar{z}}{dt}\right)^2\right\} + \Sigma m\left\{\left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + \left(\frac{dz'}{dt}\right)^2\right\}.$$

Now in whatever manner a body is moving, its motion at any instant may be considered as composed of a motion of translation of the center of gravity, and a motion of rotation round some line through the center of gravity, the direction of which will in some instances be constant, in others will be continually changing. Let $\bar{v}$ be the velocity of the center of gravity, $\frac{d\theta}{dt}$ the angular velocity of rotation, and r the distance of a particle from the line through the center of gravity round which this takes place; then the expression for the *vis viva* becomes

$$M\bar{v}^2 + \left(\frac{d\theta}{dt}\right)^2 \Sigma \delta m r^2,$$

$$\text{or } M\bar{v}^2 + Mk^2\left(\frac{d\theta}{dt}\right)^2,$$

and the equation will be

$$M\left\{\bar{v}^2 + k^2\left(\frac{d\theta}{dt}\right)^2\right\} = C + 2\int\Sigma(X dx + Y dy + Z dz).$$

It must be here carefully borne in mind, that in the general case the axis about which $\frac{d\theta}{dt}$ is estimated is continually changing its position and direction in space.

This method of expressing the *vis viva* will be of use only when we can tell, *a priori*, the direction of this axis of rotation, as in the case of motion in parallel planes; or, as it is commonly called, space of two dimensions.

The principles just proved will greatly assist us in the solution of problems. We will now proceed to explain the manner in which they should be used.

105. Suppose that the system consists of one rigid body, and that the motion is in parallel planes.

We must first write down the equations (V) with the proper forces expressed in them. We must then count how many unknown

quantities appear in them, including co-ordinates and unknown forces. Suppose the whole number to be n; then, to determine these n unknown quantities, we have the three equations (V); we must therefore have $n-3$ additional equations, expressing geometrical relations which exist between the different parts of the system.

The direct method of proceeding now is to solve these equations by any analytical artifices that suggest themselves. The unknown forces however may sometimes be eliminated very easily by means of some or all of the preceding principles.

We must therefore examine which of the above principles hold; or in other words, which of the equations expressing the above principles will be free from unknown forces. These will give us so many first integrals of the equations: that is, so many relations between the positions and velocities of the different parts of the system. Sometimes we obtain all the first integrals by means of these principles, sometimes it is necessary to combine particular artifices with them.

If we can perform the next integration the problem is completely solved, that is, the position, velocity, and direction of motion, are completely determined as functions of the time: this however can seldom be effected.

No more exact rules than these can be given. A facility in applying them to particular cases is only to be acquired by practice.

If the system consists of more than one body, we must write down the equations of motion for each separately, and then solve them by the aid of the general principles, as in the case of a single body.

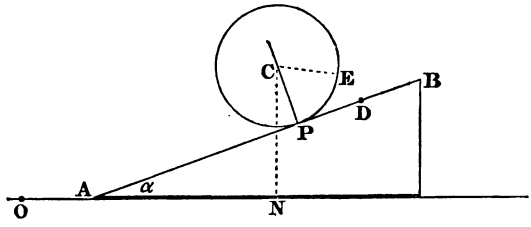
We may remark that in the cases where the general principles are applicable we might obtain the first integrals from them at once, without using the differential equations of motion of the second order. These however should always be written down.

106. We will now illustrate the application of these principles by an example of the method of determining the motion of two bodies.

A rough cylinder is placed with its axis horizontal on a rough inclined plane, which is itself moveable on a smooth horizontal plane; to determine the motion of the system.

Let a be the radius of the cylinder, M its mass; α the inclination of the plane AB, m the mass of the plane; O a fixed point in the

horizontal plane coinciding with A at the beginning of the motion, when the point E coincided with D.



$$ON = x,$$
$$NC = y,$$
$$OA = x',$$
$$PCE = \theta,$$
$$AD = l.$$

R and F the reactions of the plane on the cylinder along PC and PD. Those of the cylinder on the plane will be equal and in the opposite directions.

The equations of motion for the cylinder are

$$M\frac{d^2x}{dt^2} = F\cos\alpha - R\sin\alpha \ldots\ldots\ldots\ldots (1)$$

$$M\frac{d^2y}{dt^2} = F\sin\alpha + R\cos\alpha - Mg \ldots\ldots (2)$$

$$Mk^2\frac{d^2\theta}{dt^2} = Fa \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and for the plane

$$m\frac{d^2x'}{dt^2} = R\sin\alpha - F\cos\alpha \ldots\ldots\ldots\ldots (4).$$

These equations contain six unknown quantities, we must therefore seek for two more equations. These will express that the cylinder is in contact with the plane, and that it rolls without sliding.

The first gives

$$(x - x')\tan\alpha + a\sec\alpha = y \ldots\ldots\ldots\ldots\ldots (5).$$

The second gives

$$(x - x')\sec\alpha + a\tan\alpha + a\theta = l \ldots\ldots\ldots\ldots (6).$$

These six equations when integrated will determine the motion completely. Since we do not know whether R and F are constant or not, it is necessary to eliminate them before we can integrate.

Now in this case the principle of *vis viva* is applicable, and also the principle of the conservation of the motion of the center of gravity in a horizontal direction.

These principles therefore guide us to the method of elimination.

Adding (1) and (4) we have

$$M\frac{d^2x}{dt^2}+m\frac{d^2x'}{dt^2}=0,$$

the integral of which is

$$M\frac{dx}{dt}+m\frac{dx'}{dt}=C=0 \quad \ldots\ldots\ldots\ldots (7).$$

This is the expression of the latter of the two principles.

Again, from all the equations we have

$$M\left\{2\frac{dx}{dt}\frac{d^2x}{dt^2}+2\frac{dy}{dt}\frac{d^2y}{dt^2}+2k^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}\right\}+2m\frac{dx'}{dt}\frac{d^2x'}{dt^2}=-2Mg\frac{dy}{dt},$$

which when integrated gives

$$M\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+k^2\left(\frac{d\theta}{dt}\right)^2\right\}+m\left(\frac{dx'}{dt}\right)^2=C-2Mgy.$$

At the beginning of the motion

$$0=C-2Mg(l\sin\alpha+a\cos\alpha);$$

$$\therefore\ M\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+k^2\left(\frac{d\theta}{dt}\right)^2\right\}+m\left(\frac{dx'}{dt}\right)^2=2Mga\theta\sin\alpha \ldots (8).$$

This is the expression of the former principle.

From equations (5) and (6) we have

$$\tan\alpha\left(\frac{dx}{dt}-\frac{dx'}{dt}\right)=\frac{dy}{dt},$$

$$\text{and}\quad \sec\alpha\left(\frac{dx}{dt}-\frac{dx'}{dt}\right)+a\frac{d\theta}{dt}=0.$$

From these equations and (7) we obtain

$$\left.\begin{aligned}\frac{dx}{dt}&=-\frac{ma\cos\alpha}{M+m}\frac{d\theta}{dt},\\ \frac{dx'}{dt}&=\frac{Ma\cos\alpha}{M+m}\frac{d\theta}{dt},\\ \frac{dy}{dt}&=-a\sin\alpha\frac{d\theta}{dt},\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots (9)$$

$$\text{also}\quad k^2=\frac{a^2}{2}.$$

Substituting these values in equation (8), we have

$$\frac{3ma + Ma(1+2\sin^2\alpha)}{M+m}\left(\frac{d\theta}{dt}\right)^2 = 4g\sin\alpha\,.\,\theta,$$

the integral of which is

$$\theta = \frac{(M+m)g\sin\alpha}{3ma + Ma(1+2\sin^2\alpha)}\,t^2.$$

The constant is zero, since $\theta = 0$ when $t = 0$.

$$\text{If } A = \frac{g\sin\alpha}{3ma + Ma(1+2\sin^2\alpha)},$$

$$\theta = A(M+m)t^2;$$

$$\therefore \frac{d\theta}{dt} = 2A(M+m)t;$$

therefore, from equations (9), we have

$$\frac{dx}{dt} = -2Ama\cos\alpha\,.\,t,$$

$$\frac{dx'}{dt} = 2AMa\cos\alpha\,.\,t,$$

$$\frac{dy}{dt} = -2A(M+m)a\sin\alpha\,.\,t;$$

from which x, x' and y may be found in terms of t.

From these equations it appears that the motion both of the plane and cylinder is uniformly accelerated. Also, if we differentiate these, and substitute in equations (3) and (1) we shall obtain F and R; and it will be found that they are constant.

Again, from equations (9), we have

$$\frac{dy}{dx} = \frac{M+m}{m}\tan\alpha;$$

which shews that the center of gravity of the cylinder will descend in a right line, making with the horizontal plane an angle whose tangent is $\frac{M+m}{m}\tan\alpha$.

Instead of making use of the principle of *vis viva* to guide us in our elimination, we might have proceeded as follows: from (1), (2) and (3)

$$a\cos\alpha\frac{d^2x}{dt^2} + a\sin\alpha\frac{d^2y}{dt^2} - k^2\frac{d^2\theta}{dt^2} = -ag\sin\alpha.$$

Integrating this, we have

$$a\cos\alpha\frac{dx}{dt} + a\sin\alpha\frac{dy}{dt} - k^2\frac{d\theta}{dt} = -ag\sin\alpha\,.\,t;$$

$$\therefore a\cos\alpha\,.\,x + a\sin\alpha\,.\,y - k^2\,.\,\theta = C - \frac{1}{2}ag\sin\alpha\,.\,t^2;$$

also, the integral of (7) is

$$Mx + mx' = C',$$

from these two equations, with (5) and (6), we can find x, x', y and θ in terms of t.

There is still another method of proceeding which is always applicable in cases where the geometrical equations are linear with respect to all the variable quantities, and where the equations of motion involve the second differential coefficients only of those co-ordinates, as is the case in the present example.

From (5) and (6) we have

$$\tan\alpha\left(\frac{d^2x}{dt^2} - \frac{d^2x'}{dt^2}\right) = \frac{d^2y}{dt^2},$$

$$\text{and } \frac{d^2x}{dt^2} - \frac{d^2x'}{dt^2} + a\cos\alpha\frac{d^2\theta}{dt^2} = 0.$$

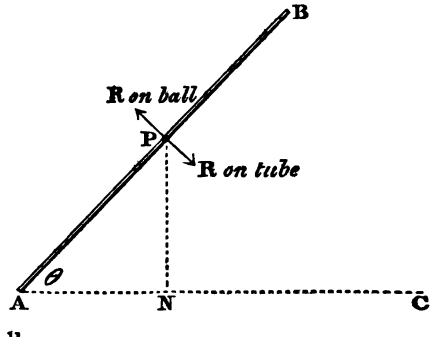
Substituting in these equations the values of the second differential coefficients given by equations (1), (2), (3) and (4), we obtain equations which involve only F, R and constant quantities.

F and R therefore are constant and may be found at once from these equations; and the equations of motion may be integrated at sight.

This is the shortest and easiest method of proceeding in cases where it is applicable.

107. As another example we will take the following problem.

In a smooth tube moveable about one extremity in a horizontal plane, a small ball is placed at a distance a from the fixed end, and an angular velocity ω is communicated to the tube: determine the motion.



Let AB be the tube of length l; P the position of the ball at time t; AC the initial position of the tube;

$$PAN = \theta,\ AN = x,$$

$$NP = y,\ AP = r,$$

M the mass of the tube, m the mass of the ball, R the reaction between tube and ball,

then for the motion of the ball we have

$$m\frac{d^2x}{dt^2} = -R\sin\theta, \ldots\ldots (1)$$

$$m\frac{d^2y}{dt^2} = R\cos\theta, \ldots\ldots\ldots (2)$$

and for the tube $$Mk^2\frac{d^2\theta}{dt^2} = -R.r. \ldots\ldots\ldots (3)$$

We have omitted two of the equations of motion of the tube, because they would introduce two additional unknown forces arising from the condition that A is fixed.

Here then we have x, y, r, θ, and R connected by three equations, we must therefore seek for two more; they are

$$x = r\cos\theta, \quad y = r\sin\theta \ldots\ldots\ldots (4)$$

Now in this case the principles of the Conservation of Areas, and of *vis viva* hold.

Conducting our eliminations accordingly we have

$$m\left\{2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{dy}{dt}\frac{d^2y}{dt^2}\right\} + 2Mk^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = 0;$$

$$\therefore m\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} + Mk^2\left(\frac{d\theta}{dt}\right)^2 = C,$$

or, transforming to polar co-ordinates,

$$m\left(\frac{dr}{dt}\right)^2 + (mr^2 + Mk^2)\left(\frac{d\theta}{dt}\right)^2 = C.$$

Now at the commencement of the motion

$$(ma^2 + Mk^2)\,\omega^2 = C;$$

$$\therefore m\left(\frac{dr}{dt}\right)^2 + (mr^2 + Mk^2)\left(\frac{d\theta}{dt}\right)^2 = (ma^2 + Mk^2)\,\omega^2. \ldots\ldots (5)$$

Again, $$m\left\{x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right\} + Mk^2\frac{d^2\theta}{dt^2} = 0.$$

From which, integrating and changing to polar co-ordinates, we have

$$(mr^2 + Mk^2)\frac{d\theta}{dt} = C,$$

at the commencement of the motion

$$(ma^2 + Mk^2)\,\omega = C;$$

$$\therefore \frac{d\theta}{dt} = \frac{ma^2 + Mk^2}{mr^2 + Mk^2}\,\omega \ldots\ldots\ldots\ldots (6)$$

Eliminating $\frac{d\theta}{dt}$ from equations (5) and (6), we have

$$\left(\frac{dr}{dt}\right)^2 = \frac{ma^2 + Mk^2}{mr^2 + Mk^2}(r^2 - a^2)\,\omega^2.$$

Hence the velocity along the tube and the angular velocity of the tube are known for any position of the particle, and depend only on its distance from A.

We cannot integrate these equations in finite terms so as to obtain the position in terms of the time.

108. Hitherto the effective force on any particle forming part of a system has been resolved in the directions of three rectangular axes fixed in space. When the motion of the particle is in one plane it is often convenient to employ polar co-ordinates, and the resolved parts of the effective force in the direction of the radius vector, and perpendicular to that direction.

The equations of motion of a single particle acted on by a force P along the radius vector from the pole, and a force T perpendicularly to that direction are

$$m\frac{d^2r}{dt^2} - mr\left(\frac{d\theta}{dt}\right)^2 = P,$$

$$2m\frac{dr}{dt}\frac{d\theta}{dt} + mr\frac{d^2\theta}{dt^2} = T.$$

Hence then from the definition of effective force if r and θ are the polar co-ordinates of a particle m forming part of a system, the effective force along the radius vector is

$$m\frac{d^2r}{dt^2} - mr\left(\frac{d\theta}{dt}\right)^2,$$

and that in a direction perpendicular to it is

$$2m\frac{dr}{dt}\frac{d\theta}{dt} + mr\frac{d^2\theta}{dt^2}.$$

Another mode of resolving the effective force is also useful, namely, in the directions of the tangent and normal of the path of the particle. These resolved parts are readily obtained as follows.

The parts parallel to the axes of x and y are $m\frac{d^2x}{dt^2}$ and $m\frac{d^2y}{dt^2}$.

Hence the part in direction of the tangent

$$= m\frac{d^2x}{dt^2}\frac{dx}{ds} + m\frac{d^2y}{dt^2}\frac{dy}{ds}$$

$$= m\frac{dt}{ds}\left(\frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{d^2y}{dt^2}\frac{dy}{dt}\right)$$

$$= m\frac{d^2s}{dt^2}.$$

And the part along the normal

$$= -m\frac{d^2x}{dt^2}\frac{dy}{ds} + m\frac{d^2y}{dt^2}\frac{dx}{ds}$$

$$= m\frac{dt}{ds}\left(\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}\right).$$

Let ρ be the radius of curvature of the path of the particle, v its velocity

$$\rho = \frac{\left(\frac{ds}{dt}\right)^3}{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}},$$

and the expression for the force along the normal becomes

$$\frac{mv^2}{\rho} \text{ or } \frac{m}{\rho}\left(\frac{ds}{dt}\right)^2.$$

When the particle moves in a circle $\frac{dr}{dt} = 0$, and the two pairs of expressions coincide with each other.

These expressions for the effective force can in any case be used instead of those referred to rectangular co-ordinates.

109. These modes of resolving the effective force, afford an explanation of a term, which, originally founded in error, has a tendency to convey an erroneous impression, viz., Centrifugal Force.

If a smooth straight tube is moved with uniform angular velocity in a horizontal plane about one extremity, and a particle is placed in it, the particle will commence moving from the fixed end of the tube, and will move with accelerated motion till it ultimately flies out of the tube.

If a particle be fastened by a string to a fixed point on a smooth horizontal table, and then be projected so as to move round this point, it will cause a tension of the string, depending on the mass of the particle and the rapidity with which it moves.

From these and similar cases, it was concluded that bodies moving round a center have a tendency to fly off from that center, and in consequence exercise a force to which the name centrifugal force was given: and its magnitude was found to be $mr\omega^2$, where r is the distance from the fixed center, and ω the angular velocity with which the body is moving round that center.

The true explanation is found in the following method of viewing it.

In the former of the preceding examples, let R be the pressure which the side of the tube exerts on the particle. This is the only impressed force on it.

Then reversing the effective forces on it, we obtain the equations of motion

$$m\frac{d^2r}{dt^2}-mr\left(\frac{d\theta}{dt}\right)^2=0, \quad \ldots\ldots\ldots(1)$$

$$R-2m\frac{dr}{dt}\frac{d\theta}{dt}-mr\frac{d^2\theta}{dt^2}=0.$$

From equation (1)

$$m\frac{d^2r}{dt^2}=mr\omega^2,$$

which shews that the motion along the tube will be the same as if the tube were at rest, and the particle were acted on by a force $mr\omega^2$ along it.

Again, in the other case, let T be the tension of the string.

Applying the effective forces reversed we have

$$-T-m\frac{d^2r}{dt^2}+mr\left(\frac{d\theta}{dt}\right)^2=0,$$

$$2m\frac{dr}{dt}\frac{d\theta}{dt}+mr\frac{d^2\theta}{dt^2}=0.$$

Now r is constant, therefore $\frac{dr}{dt}=0$, $\frac{d^2r}{dt^2}=0$; and from the latter equation $\frac{d\theta}{dt}$ is constant.

Hence the first equation becomes

$$T=mr\omega^2.$$

That is, the tension of the string is the same as it would be if the particle were at rest and acted on by a force $mr\omega^2$.

A similar result will follow in any other case in which the equations are applied to the motion of a particle, either alone or forming part of a system. So that though no such force as centrifugal force really exists, yet when a system in motion is supposed at rest, and the effective forces are applied in the opposite directions to balance the impressed forces, a portion of one of these effective forces will correspond in direction and magnitude to the fabulous centrifugal force.

Thus along the radius vector, the term $mr\left(\frac{d\theta}{dt}\right)^2$ in the expression $-m\frac{d^2r}{dt^2}+mr\left(\frac{d\theta}{dt}\right)^2$, and in the normal the whole expression $\frac{mv^2}{\rho}$ correspond to the centrifugal forces in those directions respectively.

The name "Centrifugal force," is very objectionable, from the plausibility of the reasoning by which the idea is supported. But it is found convenient in several classes of problems to have a name for the portion of the reversed effective force represented by $mr\omega^2$, and with this meaning the term centrifugal force is retained.

CHAPTER VIII.

110. In discussing the elementary principles, it was remarked that there are cases in which the whole effect produced by a force is known, when the magnitude of the force and the time during which it acts are not known; that is, when we know the definite integral $\int_{t_1}^{t_2} P dt$ without knowing either P or $t_2 - t_1$. The name "impulse" was given to this definite integral, and the force P was called an impulsive force.

Let us now consider under what circumstances forces of this sort are called into action.

We have used the term "rigid body," and have investigated the motions of rigid bodies in several cases, and have given the equations for determining that motion in all cases. Now the idea which we have attached to the term rigid is this: a body is said to be rigid when all the points preserve the same invariable mutual distances, whatever the forces which act on the different points of the body. This idea of rigidity is suggested by the first touch of external objects; though a slight examination is sufficient to convince us that it is not strictly applicable to any of them.

Different substances require different degrees of force to produce any sensible change in their form, some require very great force to do so. This approximate rigidity enables us to form the idea of the perfect quality, even though there is no substance in existence endued with that quality; and a great many substances are so nearly rigid as to convince us that calculations effected on the hypothesis of their perfect rigidity cannot be affected with sensible errors on account of the deviations from it.

In the branch of the subject however, which is now under consideration, this deviation from rigidity forms a principal feature of the problem. When pressure is exerted on a body by any means, the body slightly changes its form, or is compressed; also when the pressure ceases to act, the body wholly or partially recovers its original form. When a ball in motion strikes another ball at rest, a pressure immediately takes place between the two balls, each ball pressing on the other, and so flattening it at the point of contact. This pressure is gradually communicated from one particle to

another of each of the balls, so that velocity is communicated to the one which was at rest, and the velocity of that which was in motion is lessened; and this action continues as long as the hinder ball moves faster than the other. When the balls have acquired the same velocity, the flattening of the two is at its greatest, and if the balls had no tendency to recover their original forms the pressure between them would cease, and the two would move on together with the same velocity. Most substances however have a tendency to recover their original forms, and in consequence of this the action between the two continues till they cease to touch each other; the one which was originally at rest having the greater velocity of the two.

Now all this that we have been describing takes place in a very short time indeed: so short as to be considered instantaneous without any sensible error.

If our knowledge of the structure of matter were perfect, and if also our mathematical analysis were perfect, we should be able to investigate, from the principles already laid down, the exact nature and magnitude of the action which takes place between the two bodies, and also the degree and manner in which a change of form takes place. As it is, however, we are obliged to rest on experiment for the facts which we have stated, and for the laws which we are going to state.

Let t_1 be the time when the first contact takes place, t_2 the time when the two bodies are moving with the same velocity, and t_3 the time when they cease to touch at all. Then the whole action which takes place during compression, as it is termed, will be represented by $\int_{t_1}^{t_2} P\,dt$ and the whole action during restitution (that is, during the recovery of the original form) by $\int_{t_2}^{t_3} P\,dt$.

The experimental law which forms the basis of our calculations is that

$$\int_{t_2}^{t_3} P\,dt = e\int_{t_1}^{t_2} P\,dt;$$

where e is a multiplier less than unity, which is constant for all values of P, so long as the substances between which the impact takes place remain unchanged.

The less or the greater the change of form that takes place for a given value of $\int_{t_1}^{t_2} P\,dt$, the harder or softer the body is said to be.

The greater or less e is, the greater or less is said to be the "elasticity" of the bodies.

If $e = 1$, the elasticity is said to be perfect.

e is called the coefficient of elasticity, or sometimes the elasticity. The definite integrals are generally written R and R', so that $R' = eR$.

R and R' are called the "impulses" during compression and restitution, and $R + R'$, or $(1 + e) R$ is called the whole impulse.

The law has been enunciated and the action between the bodies explained in a particular case only; it is however applicable, with a very slight extension, to all cases.

111. Since an impulsive force is only a force of great intensity, D'Alembert's principle is applicable to it, as to all other forces.

If we take the case of one rigid body, that is, rigid in the modified sense in which we have explained the term, the equations of motion will be the same as before, and may, as in that case, be reduced to

$$\left.\begin{aligned} M\frac{d^2\bar{x}}{dt^2} &= \Sigma X, \\ M\frac{d^2\bar{y}}{dt^2} &= \Sigma Y, \\ M\frac{d^2\bar{z}}{dt^2} &= \Sigma Z. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \text{(III)}$$

$$\text{And } \left.\begin{aligned} \Sigma\left\{y\left(Z - m\frac{d^2z}{dt^2}\right) - z\left(Y - m\frac{d^2y}{dt^2}\right)\right\} &= 0, \\ \Sigma\left\{z\left(X - m\frac{d^2x}{dt^2}\right) - x\left(Z - m\frac{d^2z}{dt^2}\right)\right\} &= 0, \\ \Sigma\left\{x\left(Y - m\frac{d^2y}{dt^2}\right) - y\left(X - m\frac{d^2x}{dt^2}\right)\right\} &= 0. \end{aligned}\right\} \ldots \text{(IV)}$$

Now in this case, ΣX, ΣY and ΣZ include all the forces which act on the body, impulsive or otherwise; and if we take the definite integrals of the first three equations between the times t_1 and t_2, the beginning and end of impact, we have

$$M(u_2 - u_1) = \Sigma \int_{t_1}^{t_2} X dt,$$

$$M(v_2 - v_1) = \Sigma \int_{t_1}^{t_2} Y dt,$$

$$M(w_2 - w_1) = \Sigma \int_{t_1}^{t_2} Z dt,$$

or using X', Y', Z' to denote the "Impulses,"

$$\left.\begin{aligned} M(u_2-u_1) &= \Sigma X', \\ M(v_2-v_1) &= \Sigma Y', \\ M(w_2-w_1) &= \Sigma Z'. \end{aligned}\right\} \ldots\ldots\ldots\ldots \text{(VI)}$$

The limits t_1 and t_2 must be taken so as to include the whole of every impulse, and consequently may include a longer time than is occupied by some of them. This however will produce no error; for suppose one of the impulses as X' to commence at t_1' and end at t_2' both of which times are between t_1 and t_2.

Then since

$$\int_{t_1}^{t_2} X dt = \int_{t_1}^{t_1'} X dt + \int_{t_1'}^{t_2'} X dt + \int_{t_2'}^{t_2} X dt,$$

and since X is zero between t_1 and t_1', and also between t_2' and t_2,

$$\text{we have } \int_{t_1}^{t_2} X dt = \int_{t_1'}^{t_2'} X dt = X'.$$

Also since $\int_{t_1}^{t_2} X dt$ has a sensible magnitude in consequence of X being very large; this integral will not have a sensible magnitude for those forces which are not impulsive; in other words, these forces will not produce a sensible effect on the velocity during the short time of impact, and therefore do not appear in equations (VI).

In the same way we may integrate equations (IV) between t_1 and t_2; and since the interval $t_2 - t_1$ is extremely short x, y, z, &c., may be considered constant during that interval, so that we have

$$\left.\begin{aligned} \Sigma\{y[Z'-m(w'-w)]-z[Y'-m(v'-v)]\} &= 0, \\ \Sigma\{z[X'-m(u'-u)]-x[Z'-m(w'-w)]\} &= 0, \\ \Sigma\{x[Y'-m(v'-v)]-y[X'-m(u'-u)]\} &= 0, \end{aligned}\right\} \ldots \text{(VII)}$$

In these equations x, y, z are referred to the center of gravity as origin, u is the value of $\frac{dx}{dt}$ before impact, and u' its value after impact, and similarly for the other letters.

(VI) and (VII) give the state of motion of the body immediately after impact, without any integration beyond that expressed by Σ.

When the motion of every particle before and after impact is parallel to one plane, equations (VII) are reduced to the one equation

$$\Sigma m\{x(v'-v)-y(u'-u)\} = \Sigma(xY'-yX').$$

Let ω_1 and ω_2 be the angular velocities of the body before and after impact, then

$$u = -y\omega_1, \quad u' = -y\omega_2, \quad v = x\omega_1, \quad v' = x\omega_2.$$

Let L' be the resultant moment of the impulses, and let $r^2 = x^2 + y^2$, then this equation becomes

$$\Sigma m r^2 (\omega_2 - \omega_1) = L'.$$

And the three equations in this case are

$$\left.\begin{aligned} M(u_2 - u_1) &= \Sigma X', \\ M(v_2 - v_1) &= \Sigma Y', \\ Mk^2(\omega_2 - \omega_1) &= L'. \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(VIII)}$$

112. If the system consists of more than one body, the equations must be written down for each body separately, and the unknown forces which enter the equations, must be determined by the relations existing between the velocities of the points at which impact takes place. These will be different according as the bodies are elastic or inelastic, and as their surfaces are rough or smooth.

When the bodies are inelastic and smooth, the whole impulse will be in the direction of the normal to the surface at the point of contact, and must be determined from the condition that the resolved parts in the direction of this normal of the velocities of the points in contact must be equal. If the surfaces are rough, the whole velocities of the two points will be equal and in the same direction.

When the bodies are elastic, and the coefficient of elasticity is given, the normal impulse may be found from the consideration that the normal velocities of the points in contact are the same at the end of compression, and that the impulse during restitution is in a constant ratio to the impulse during compression: when the surfaces are rough, the velocities of the points in contact resolved along the tangent plane will be equal at the conclusion of the impact, while the resolved parts along the normal will be equal only at the end of compression.

If the surfaces are partially rough, as is always the case in nature, let μ be the coefficient of friction and R the normal portion of the impulsive force; then the tangential portion of the impulsive force cannot exceed μR, and therefore the whole tangential impulse cannot exceed $\int_{t_1}^{t_2} \mu R\, dt$.

Little is known experimentally of the value of μ for forces so great as to be impulsive; so far as the experiments go, however,

they indicate a diminution of μ for large values of the normal pressure. Considering μ to be approximately constant, we have

$$\int_{t_1}^{t_2} \mu R\,dt = \mu \int_{t_1}^{t_2} R\,dt = \mu R',$$

where R' is the normal impulse. This is the greatest amount of tangential impulse that can take place for a given amount of normal impulse; all will not necessarily be exerted in any particular instance. The preceding remarks will be better understood by a consideration of the following examples.

113. Let a ball whose mass is m_1 moving with a velocity v_1 overtake a ball whose mass is m_2 moving with a velocity v_2, and let e be the coefficient of elasticity.

Let R be the impulse during compression, u the common velocity of the two balls after compression, u_1 and u_2 their velocities after impact: then from equations (VIII),

$$m_1(u - v_1) = -R,$$
$$m_2(u - v_2) = R,$$

from which we have $u = \dfrac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$,

and $R = \dfrac{m_1 m_2 (v_1 - v_2)}{m_1 + m_2}$,

the whole impulse is $(1 + e)R$, and the velocities after impact are given by the equations

$$m_1(u_1 - v_1) = -(1 + e)\frac{m_1 m_2 (v_1 - v_2)}{m_1 + m_2},$$

$$m_2(u_2 - v_2) = (1 + e)\frac{m_1 m_2 (v_1 - v_2)}{m_1 + m_2}.$$

If the balls are equal and the elasticity perfect, these equations become $u_1 = v_2$, and $u_2 = v_1$.

114. Again, let a sphere of radius a revolving with the angular velocity ω about a horizontal axis fall vertically on a horizontal plane; and let μ and e be the coefficients of friction and elasticity between the sphere and the plane.

Suppose, to fix the ideas, that the axis of x is horizontal, and the plane of xy perpendicular to the axis of rotation.

Let M be the mass of the sphere, V the velocity of its center of gravity before impact; v_x, v_y the resolved parts of the velocity, and

ω' the angular velocity after compression; u_x, u_y and ϖ the same quantities when the impact is concluded; R and F the normal and tangential impulses during compression; then

$$Mv_x = F \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$M(v_y + V) = R \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$Mk^2(\omega' - \omega) = Fa \quad \ldots\ldots\ldots\ldots\ldots (3)$$

and the condition that the point of contact is at rest gives

$$v_x + \omega' a = 0, \qquad v_y = 0 \quad \ldots\ldots\ldots\ldots\ldots (4)$$

we also have the condition that F cannot exceed μR.

From equations (2) and (4), we have

$$R = MV,$$

also from (1), (3) and (4), we have

$$\frac{F}{M} + \omega a + \frac{Fa^2}{Mk^2} = 0;$$

$$\therefore F = -\frac{Mak^2\omega}{a^2 + k^2},$$

provided this is not greater than MV. Let this be the case, then if F' is the tangential impulse during restitution

$$Mu_x = F + F' \quad \ldots\ldots\ldots\ldots\ldots (5)$$

$$M(u_y + V) = (1 + e)R \quad \ldots\ldots\ldots\ldots\ldots (6)$$

$$Mk^2(\varpi - \omega) = (F + F')a \quad \ldots\ldots\ldots\ldots\ldots (7)$$

with the condition $u_x + \varpi a = 0 \quad \ldots\ldots\ldots\ldots\ldots (8)$

Equation (6) gives $u_y = eV$,

also from (5), (7), and (8)

$$F + F' = -\frac{Mak^2\omega}{a^2 + k^2},$$

and therefore $F' = 0$, and from (5) and (7)

$$u_x = -\frac{ak^2\omega}{a^2 + k^2} \text{ and } \varpi = \frac{k^2\omega}{a^2 + k^2},$$

and the initial motion after impact is completely determined.

We may remark here that the whole change of angular velocity took place during compression: if $\frac{ak^2\omega}{a^2 + k^2}$ had been greater than V but less than $(1 + e)V$, part of this change would have taken place during restitution, and the final result would have been the same.

Let $\frac{ak^2\omega}{a^2+k^2}$ be greater than $(1+e)V$; in this case instead of the equation $u_x + \varpi a = 0$, we have

$$F + F' = -\mu(1+e)MV;$$

$$\therefore\ u_x = -\mu(1+e)V \text{ and } \varpi = \omega - \frac{\mu(1+e)Va}{k^2},$$

and $u_y = eV$ as before.

115. As a concluding example we will take the system described in Art. 106, and preserve the same notation.

Let the cylinder fall from rest so as to strike the plane along a horizontal line.

Let V be the velocity of the cylinder before impact; v_x, v_y the resolved parts of the velocity of its center of gravity after compression, ω its angular velocity, v the velocity of the plane at the same time; u_x, u_y, ϖ and u the same quantities after restitution.

Then we have

$$Mv_x = F\cos\alpha - R\sin\alpha,$$
$$M(v_y + V) = F\sin\alpha + R\cos\alpha,$$
$$Mk^2\omega = Fa,$$
$$mv = R\sin\alpha - F\cos\alpha.$$

Since the velocities of the points of the two bodies in contact are the same, their resolved parts in any two directions will be the same, therefore

$$v = v_x + a\omega\cos\alpha,$$
$$0 = v_y + a\omega\sin\alpha,$$

from which $R = MV\cos\alpha\,\frac{M+3m}{M(1+2\sin^2\alpha)+3m}$.

After restitution the equations are

$$Mu_x = F'\cos\alpha - (1+e)R\sin\alpha,$$
$$M(u_y + V) = F'\sin\alpha + (1+e)R\cos\alpha,$$
$$Mk^2\varpi = F'a,$$
$$mu = (1+e)R\sin\alpha - F'\cos\alpha,$$

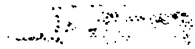
and the condition in this case is that the velocities along the plane shall be the same, which gives the equation

$$(u_x - u)\cos\alpha + u_y\sin\alpha + a\varpi = 0.$$

R is already known, and from these equations we can determine F' and the other unknown quantities.

116. An examination of the proof of the Principles of the Conservation of Areas and of the Motion of the center of gravity, will shew that these principles are true when the forces are impulsive, in the same cases that they are true when the forces are finite. The Principle of the Conservation of *vis viva* on the contrary is not so. In fact the idea of an impulse implies a change in the geometrical relations of the system.

These principles, however, are of little use in the solution of problems where the forces are impulsive, since the equations of motion are all linear, and require no integration.




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